PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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Phil. Trans. R. Soc. Lond. A 1997 **355**, 869-919
doi: 10.1098/rsta.1997.0049

Equilibrium and mutual attraction or repulsion of objects supported by surface tension

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Phil. Trans. R. Soc. Lond. A (1997) **355**, 869–919
Printed in Great Britain

TEX Paper

A theory is presented to explain known phenomena associated with lightweight objects on the surface of a liquid, and to predict other phenomena. Such objects are supported partly by hydrostatic pressure and partly by surface tension, and this latter component causes the surface of the liquid to deflect in a manner that decays rapidly with distance from each object. The forces between such objects, which may be of attraction or repulsion, stem from the interaction of these surface deflexions and are here determined by reference to the equal and opposite forces required to maintain static equilibrium. As an essential preliminary to this, and in the context of a linear theory, the equilibrium equations are derived for a floating object of arbitrary shape where the deflexions and slopes of the adjoining free surface of the liquid are also arbitrary but small. These equilibrium equations provide the boundary conditions that determine the deflexion of the free surface of the liquid based on Laplace's surface tension equation. The forces of mutual attraction or repulsion are shown to be given by certain contour integrals involving the squares of the surface deflexion and slopes surrounding each object.

One-dimensional cases of infinite strips supported on an infinite expanse of liquid are considered in detail because, first, they admit of exact, nonlinear solutions so that the range of validity of the linear theory may be estimated and, second, they demonstrate in a simple manner many of the phenomena associated with objects supported by surface tension, including: mutual attraction leading to coalesence, characteristic deflexion patterns in rafts dependent on the individual strip width, localized mutual repulsion between objects of different weights, and extensive mutual repulsion between an object and a boundary. It is also shown that a phenomenon of mutual alignment occurs with certain strips with pie-crust edge undulations.

An analysis is given of the toppling instability of an upright circular cylinder, and its equilibrium state if its centre of gravity is radially offset; we also outline an inverse method of analysis for determining the forces of mutual attraction between discs of arbitrary shape, and the method is demonstrated for two touching oval discs. Finally, attention is given to the forces of mutual attraction within and between coalescences of numerous objects whose individual linear dimensions are small in comparison with the capillary length.

Various results are supported by experiment.

1. Introduction

The phenomenon of surface tension manifests itself in many ways; these include capillary action in liquids that rise in a narrow tube, the behaviour of mercury on a horizontal surface, and the formation of raindrops and soap bubbles. It is also responsible in an indirect manner for the commonly observed, but hitherto unexplained, phenomenon that small lightweight objects in close proximity on the surface of a liquid will generally coalesce.

The theory of capillary action based on surface tension was founded by Young (1805) but because of his scrupulous avoidance of mathematics his results are better understood by reference to the work of Laplace (1806). The aspect of surface tension that these authors deduced, and which is of prime concern to us, is that at the free surface of a liquid that is in static equilibrium the hydrostatic pressure is balanced by the product of the surface tension and the total curvature of the free surface. In general, the analysis for determining such surfaces presents formidable difficulties while, even when there is rotational symmetry, Lord Kelvin (1891) wrote: 'A general solution of this problem by the methods of the differential and integral calculus transcends the powers of mathematical analysis'. Nowadays, however, we can readily obtain numerical solutions for such problems as the shape of a drop of liquid resting on a flat surface (Brown *et al.* 1980) or the effects of capillary action in a cylindrical container (Adamson 1990). There is another class of problem where numerical or analytical solutions are readily obtainable because the governing equations are linear; these occur if the slope of the free surface relative to a horizontal x, y plane is known to be small, for then its curvatures are adequately given by the second derivatives with respect to x, y of its vertical displacement. This procedure yields Laplace's equation which is associated with a characteristic length L that depends on the magnitude of the surface tension and the weight density of the liquid, for example $L \approx 2.7$ mm for water. Laplace's equation forms an essential ingredient in the present theory and it follows that solutions for objects of a given size on the surface of a given liquid are not applicable at a different scale. Furthermore, because of the rapid decay of surface deflexion with distance from an object, the importance of surface tension effects is generally limited to objects separated by a distance less than about $10L$. As for the forces of mutual attraction or repulsion between objects, we derive these by equating them with the equal and opposite forces required for static equilibrium. Thus, dynamic effects are not considered and the theory is strictly applicable only to objects that have coalesced or, as may occur, have reached an equilibrium position where they remain separated. However, the theory will necessarily form part of any dynamic analysis, in addition to providing initial 'at rest' conditions. The forces of mutual attraction or repulsion applied to each object are expressed in terms of certain contour integrals of the squares of the surface deflexion and slopes surrounding each object. However, to enable us to estimate the range of validity of the linear theory, we first derive exact (nonlinear) solutions for the one-dimensional problems of a single and two touching infinitely long, shallow, rigid strips of rectangular section supported on the surface of an infinite expanse of liquid. For the single strip we also discuss the basic role played by the physical characteristics of the object, liquid and gas, and we determine the critical weight at which the strip sinks. We also show how linear theory can give a safe estimate of the critical weight at which any object will sink, despite the fact that this is an essentially nonlinear phenomenon.

2. Range of validity of linear theory

Before deriving the general equilibrium equations for an arbitrary object based on linear theory, we investigate its range of validity by comparisons with exact (nonlin-

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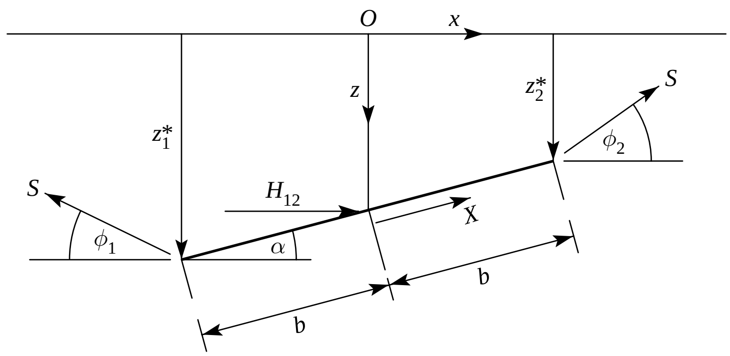

Figure 1. Cross-section of shallow strip with arbitrary edge displacements z^* and slopes ϕ of the liquid surface.

ear) solutions for the mathematically one-dimensional problems of a single infinitely long strip of shallow rectangular cross-section and of two identical but touching strips supported on the surface of an infinite expanse of liquid. We also discuss the role played by the physical characteristics of the object, liquid and gas. First, we rehearse the exact (nonlinear) equation for the static deflexion of a liquid and its specialization to the one-dimensional case. We then derive the boundary conditions for a strip and an expression for the horizontal force required for its equilibrium. The corresponding results for linear theory are derived by a limiting process.

The equation governing the surface deflexion z of a liquid with surface tension S per unit length is given by

$$S(\kappa_1 + \kappa_2) - \gamma g z = 0, \tag{2.1}$$

where

$$\begin{aligned} \kappa_1, \kappa_2 &= \text{principal curvatures of surface,} \\ \gamma &= \text{density of liquid,} \\ g &= \text{gravitational acceleration.} \end{aligned}$$

For the one-dimensional case of a strip of liquid aligned to the y-axis, say, equation (2.1) is therefore given by

$$S\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}\left\{1 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2\right\}^{-3/2} - \gamma g z = 0. \tag{2.2}$$

(a) *General equilibrium conditions for shallow strip*

The equilibrium conditions for a shallow strip of rectangular cross-section are now expressed in terms of arbitrary slopes ϕ_1, ϕ_2 and deflexions z_1^*, z_2^* of the free surface of the liquid at its junction with the strip, as shown in figure 1.

The strip is supported by surface tension and hydrostatic pressure, and hence vertical equilibrium yields

$$S(\sin\phi_1 + \sin\phi_2) + \gamma g b(z_1^* + z_2^*)\cos\alpha = W, \tag{2.3}$$

where W is the weight of the strip per unit length, $2b$ is the width of the strip and α is the angle of tilt of the strip cross-section, so that

$$\sin\alpha = \left(\frac{z_1^* - z_2^*}{2b}\right). \tag{2.4}$$

Likewise, horizontal equilibrium yields

$$H_{12} = S(\cos\phi_1 - \cos\phi_2) + \gamma g b(z_1^* + z_2^*)\sin\alpha, \tag{2.5}$$

where the suffices for H indicate the direction of the horizontal force required for equilibrium.

In discussing moment equilibrium it is convenient to regard the horizontal force as having two components, namely $\frac{1}{2}(1+\psi)H_{12}$ at edge$_1$ and $\frac{1}{2}(1-\psi)H_{12}$ at edge$_2$, because this facilitates the treatment of cases in which strips have coalesced when, for example, $\psi = \pm 1$ for the outermost strips. Moments about the centre-line of the lower surface of the strip now yields

$$bS\{\sin(\phi_1+\alpha) - \sin(\phi_2-\alpha)\} - b\psi H_{12}\sin\alpha - \int_{-b}^{b} \gamma g z X \,\mathrm{d}X = 0, \tag{2.6}$$

where X is measured across the width of the deflected strip. Now the deflexion z varies linearly from z_1^* to z_2^* and hence the integral in equation (2.6) is given by

$$-\int_{-b}^{b} \gamma g z X \,\mathrm{d}X = \tfrac{1}{3}\gamma g b^2(z_1^* - z_2^*). \tag{2.7}$$

(*b*) *Non-dimensionalization of lengths*

The above equations can be expressed more simply by the introduction of the following non-dimensional measures of the planar coordinates x, y, the surface deflexion z, the outward normal n (measured in a horizontal plane) from the edge of a strip, the half width b of a strip and, to be introduced later, the distance between adjacent strips l:

$$\{\xi, \eta, \zeta, \nu, \beta, \lambda\} = \{x, y, z, n, b, l\}/L, \tag{2.8}$$

where

$$L = (S/\gamma g)^{1/2}, \tag{2.9}$$

which is sometimes referred to as the *capillary length* or the *surface tension length*. Thus, in non-dimensional form, the exact differential equation for the surface of the liquid in the one-dimensional case is given by

$$\frac{\mathrm{d}^2\zeta}{\mathrm{d}\xi^2}\left\{1 + \left(\frac{\mathrm{d}\zeta}{\mathrm{d}\xi}\right)^2\right\}^{-3/2} - \zeta = 0, \tag{2.10}$$

while the equation of vertical equilibrium is given by

$$\sin\phi_1 + \sin\phi_2 + \beta(\zeta_1^* + \zeta_2^*)\cos\alpha = W/S, \tag{2.11}$$

where

$$\sin\alpha = \left(\frac{\zeta_1^* - \zeta_2^*}{2\beta}\right). \tag{2.12}$$

The equation of horizontal equilibrium is given by

$$H_{12}/S = \cos\phi_1 - \cos\phi_2 + \tfrac{1}{2}(\zeta_1^{*2} - \zeta_2^{*2}), \tag{2.13}$$

and, after substitution of the above expression into equation (2.6), the equation of moment equilibrium is given by

$$\sin\alpha\{(1-\psi)\cos\phi_1 + (1+\psi)\cos\phi_2\} + \cos\alpha(\sin\phi_1 - \sin\phi_2) \\ + \beta\{\tfrac{1}{3}(\zeta_1^* - \zeta_2^*) - \psi\sin^2\alpha(\zeta_1^* + \zeta_2^*)\} = 0. \tag{2.14}$$

Note that because we are concerned here only with *static* equilibrium, the horizontal force H_{12} cannot be initially specified as it is a function of the edge slopes ϕ and displacements ζ^* which are determined by the boundary conditions of equations (2.11) and (2.14).

(*c*) *Linear theory*

Linear theory is based on the assumption that the squares of the slopes of the surface of the liquid and the angle of tilt of the object, may be neglected in comparison with unity. With this simplification equation (2.1) becomes

$$\nabla^2\zeta - \zeta = 0, \tag{2.15}$$

while for the problem with strips aligned to the η axis,

$$\frac{\mathrm{d}^2\zeta}{\mathrm{d}\xi^2} - \zeta = 0. \tag{2.16}$$

Likewise, the boundary conditions of equations (2.11) and (2.14) reduce to

$$\left.\begin{aligned} \phi_1 + \phi_2 + \beta(\zeta_1^* + \zeta_2^*) &= W/S, \\ \beta(\phi_1 - \phi_2) + (1 + \tfrac{1}{3}\beta^2)(\zeta_1^* - \zeta_2^*) &= 0, \end{aligned}\right\} \tag{2.17}$$

and the horizontal force required for equilibrium is given by

$$H_{12}/S = \tfrac{1}{2}(\phi_2^2 - \phi_1^2 + \zeta_1^{*^2} - \zeta_2^{*^2}), \tag{2.18}$$

where

$$\phi = -\left(\frac{\partial\zeta}{\partial\nu}\right)_{\nu=0}. \tag{2.19}$$

Note that in the equation of moment equilibrium the term ψ no longer appears. This is because the horizontal force varies as terms of order ϕ^2 and $\alpha\zeta^*$, and its contribution to moment equilibrium varies as terms of order $\alpha\phi^2$ and $\alpha^2\zeta^*$ which are negligible in comparison with other terms.

(*d*) *Equilibrium of single strip*

For the exact theory, the reader may care to note that equation (2.10) is the same as that for the elastica which was comprehensively treated by Kelvin & Tait (1895). Thus, multiplication by $2\mathrm{d}\zeta$ and integration gives

$$\zeta^2 = C - 2\left\{1 + \left(\frac{\mathrm{d}\zeta}{\mathrm{d}\xi}\right)^2\right\}^{-1/2}, \tag{2.20}$$

where C is a constant. Now as $\xi \to \infty$, $\zeta \to 0$ so that $C = 2$ and it follows that equation (2.20) can be expressed in the form

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\xi} = -\frac{\zeta(4-\zeta^2)^{1/2}}{2-\zeta^2}. \tag{2.21}$$

Because of symmetry the boundary conditions at each edge are the same, where we therefore have

$$\tan\phi = \frac{\zeta^*(4-\zeta^{*^2})^{1/2}}{2-\zeta^{*^2}}, \tag{2.22}$$

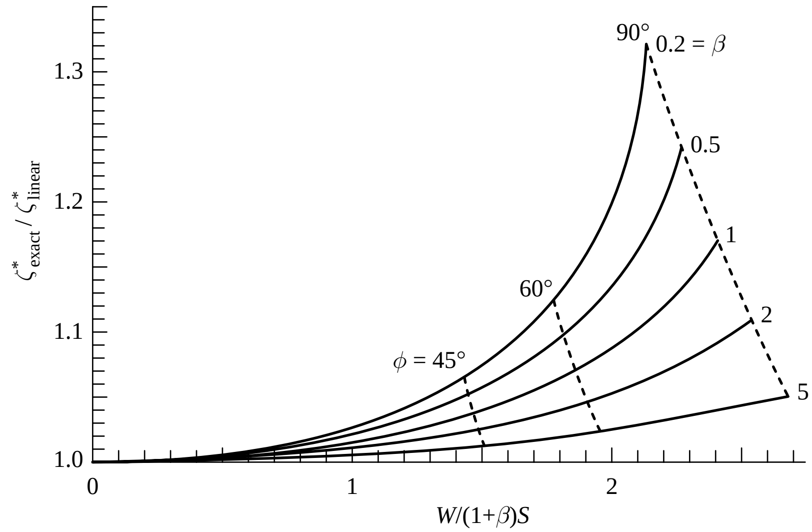

Figure 2. Variation of $\zeta^*_{\text{exact}}/\zeta^*_{\text{linear}}$ with $W/(1+\beta)S$.

so that, in particular,

$$\sin\phi = \zeta^*(1 - \tfrac{1}{4}\zeta^{*^2})^{1/2}, \qquad \cos\phi = 1 - \tfrac{1}{2}\zeta^{*^2}. \tag{2.23}$$

It follows from the boundary condition of vertical equilibrium, equation (2.11), that the deflexion of the strip is determined by

$$\zeta^* = \frac{W}{2S}\left(\frac{1}{\beta + (1 - \frac{1}{4}\zeta^{*2})^{1/2}}\right). \tag{2.24}$$

The strip deflexion thus varies nonlinearly with W, in contrast to linear theory which yields

$$\zeta^* = \frac{W}{2S}\left(\frac{1}{1+\beta}\right), \qquad \phi = \zeta^*. \tag{2.25}$$

A compact comparison of the deflexions according to the exact and linear theories is shown in figure 2 where, for various values of β, $\zeta^*_{\text{exact}}/\zeta^*_{\text{linear}}$ is plotted against $W/(1+\beta)S$. Also shown are curves along which the edge slope $\phi = 45$, 60 and 90°. Note that when $\phi = 90°$ the strip remains stable because a small increase in W is equilibrated by an increase in hydrostatic pressure, i.e.

$$\left(\frac{\partial W}{\partial z^*}\right)_{\phi=\frac{1}{2}\pi} = 2b\gamma g. \tag{2.26}$$

For a given width of strip the maximum possible load that can be supported occurs when $\partial W/\partial z^* = 0$, whence from equation (2.24),

$$W_{\max} = \tfrac{1}{2}S\{4 + 10\beta^2 - \tfrac{1}{2}\beta^4 + \tfrac{1}{2}\beta(8+\beta^2)^{3/2}\}^{1/2}, \tag{2.27}$$

which occurs when

$$\zeta^* = \{2 - \tfrac{1}{2}\beta^2 + \beta(2 + \tfrac{1}{4}\beta^2)^{1/2}\}^{1/2}, \qquad \phi = \tfrac{1}{2}\pi + \arcsin\left(\frac{2\beta}{\beta + (8+\beta^2)^{1/2}}\right). \tag{2.28}$$

Of course, such extreme situations have little practical relevance but are presented for completeness; furthermore, because $\phi > 90°$ it would be necessary with a strip

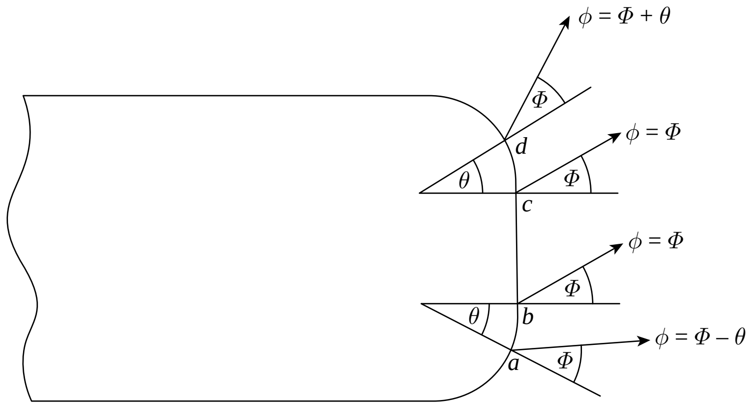

Figure 3. Cross-section of edge of strip with rounded corners. Points a, b, c, d indicate contact with surface of liquid for increasing weights of strip. Arrows indicate direction of surface tension.

of rectangular section for the free surface of the liquid to coincide with the *upper* edges of the strip so that the above value of $W_{\max}$ would be increased by buoyancy effects, as discussed below where attention is drawn to certain restrictions on the range of validity of equation (2.27). Figure 2 shows, however, that linear theory gives excellent agreement if $\phi < 45°$, while even if $\phi = 60°$ the maximum error is generally less than 10%, despite the basic assumption that ϕ is small.

(*e*) *Influence of physical characteristics of object, liquid and gas*

The analysis of § 2 *a* is based on the assumption that the free surface of the liquid is in contact with a strip only at its lower edges. This assumption is valid for sufficiently small values of ϕ, but here we determine its precise range of validity and what happens when this range is exceeded. In this connection it is convenient to introduce Φ, say, the angle in a vertical plane between a normal to the surface of the object and the free surface of the liquid. The angle Φ depends on physical characteristics of the object, liquid and gas, and is governed by Young's equation, namely

$$S \sin \Phi = S_{\mathrm{sl}} - S_{\mathrm{sg}}, \tag{2.29}$$

where S_{sl}, S_{sg} are, respectively, the interfacial tensions between the solid and liquid, and the solid and gas. In contrast, the angle ϕ between the horizontal and the free surface of the liquid depends primarily on considerations of equilibrium and in what follows we determine the influence of Φ on ϕ for the single uniform strip of rectangular section.

Figure 3 depicts cross-sections of differently weighted strips of rectangular section with, for demonstration purposes, rounded corners. The arrowed lines are tangents to the surface of the liquid. For rectangular sections with no rounded corners the points a, b coalesce, as do the points c, d. Furthermore, it is apparent that when $\phi < \Phi$ the free surface of the liquid coincides with the lower edges of the strip, while if $\phi > \Phi$ it coincides with the upper edges of the strip. Finally, if $\phi = \Phi$ the surface of the liquid meets the sides at some intermediate point, see below, where the three cases are analysed by the linear theory.

(i) *Liquid surface coinciding with lower edges of strip, $\phi < \Phi$*

This case was treated in § 2 *d* and it only remains to identify the weight limit of the strip or, what is more convenient, the upper limit of the density γ_{s} of a strip of given cross-section $2b \times 2h$. It follows from equations (2.25) that the surface of the liquid coincides with the lower edges of the strip if

$$\gamma_{\mathrm{s}} \leqslant \Psi\Phi, \tag{2.30}$$

where

$$\Psi = \frac{S(1+\beta)}{2bhg}.$$

(ii) *Liquid surface coinciding with upper edges of strip, $\phi > \Phi$*

This case is effectively identical to that considered above except that, from considerations of buoyancy, the strip has an effective density of $(\gamma_{\mathrm{s}} - \gamma)$. It follows that the lower limit for the density is given by:

$$\gamma_{\mathrm{s}} \geqslant \Psi\Phi + \gamma. \tag{2.31}$$

(iii) *Liquid surface meeting side of strip, $\phi = \Phi$*

Suppose that the free surface of the liquid meets the sides of the strip at a height $2\Delta h$ above the lower edges, so that $0 < \Delta < 1$. The effective density of the strip is now $(\gamma_{\mathrm{s}} - \Delta\gamma)$ and it follows that

$$\gamma_{\mathrm{s}} = \Psi\Phi + \Delta\gamma, \tag{2.32}$$

so that Δ varies linearly from 0 to 1 as γ_{s} varies between the limiting values of $\Psi\Phi$ and $(\Psi\Phi + \gamma)$.

(iv) *Influence of Φ on theoretical maximum load*

The analysis leading up to equation (2.27) does not include any restriction imposed by the physical characteristics of the object, liquid and gas. Thus from equation (2.28) it follows that equation (2.27) is valid if

$$\Phi \leqslant \arcsin\left(\frac{2\beta}{\beta + (8+\beta^2)^{1/2}}\right), \tag{2.33}$$

which can be expressed as

$$\beta \leqslant \sin\Phi\left(\frac{2}{1-\sin\Phi}\right)^{1/2}. \tag{2.34}$$

When this inequality is not satisfied the maximum load occurs when $\phi = \frac{1}{2}\pi + \Phi$, whence from equations (2.23), (2.24) and (2.31),

$$W_{\max} = 2S[\cos\Phi + \beta\{2(1+\sin\Phi)\}^{1/2}] + 4bh\gamma g. \tag{2.35}$$

The single uniform strip is, needless to say, the simplest case amenable to such analysis. For example, further, albeit minor, complexities arise if the liquid surface meets the side of a tilted strip. For more general objects it may be necessary to analyse by numerical trial-and-error techniques.

(v) *Discussion*

Linear theory cannot predict the critical weight at which a strip or, indeed, any object will sink because this is an essentially nonlinear phenomenon. However, it follows from the previous exact solutions that a strip will be slightly below that critical weight when $\phi = \frac{1}{2}\pi$, particularly if Φ is small. Indeed, referring to equation

(2.3) we see that the component of support provided by surface tension is $2S\sin\phi$ whose maximum value occurs when $\phi = \frac{1}{2}\pi$. In linear theory whose validity, strictly speaking, requires that ϕ is small, the corresponding component of support is approximated as $2S\phi$, so that to avoid exceeding the true maximum value we see that ϕ should not exceed 1. At this limiting value of ϕ the component of support provided by surface tension agrees with the true maximum value, but the component provided by hydrostatic pressure is underestimated because linear theory then predicts that $\zeta^* = 1$, whereas the exact value is $\sqrt{2}$. For strips that are supported primarily by surface tension this discrepancy is less significant; it also means, however, that a strip whose weight is such that linear theory predicts a unit value of ϕ will be somewhat below that required for it to sink. Similar arguments apply to the behaviour of circular discs, and these can doubtless be extended to more general objects. Thus, although linear theory cannot predict the precise conditions at which an object will sink, it can predict a weight that is known to be not too far below the critical value. Of course, in any biological application concerning, for example, the equilibrium or motion of surface-living insects or their eggs, natural selection would have ensured that they remain afloat under moderate conditions of surface disturbance, and hence the requirements of linear theory are likely to be met.

(*f*) *Equilibrium of two identical, shallow, touching strips*

The deflexions of the strips are symmetrical about their line of contact and hence we can focus attention on the right-hand strip, say, where $\phi_1 = 0$ and $\psi = 1$. Equation (2.11) of vertical equilibrium is thus given by

$$\sin\phi_2 + \beta(\zeta_1^* + \zeta_2^*)\cos\alpha = W/S, \tag{2.36}$$

and equation (2.14) of moment equilibrium reduces to

$$2\sin\alpha\cos\phi_2 - \cos\alpha\sin\phi_2 + \beta\{\tfrac{1}{3}(\zeta_1^* - \zeta_2^*) - (\zeta_1^* + \zeta_2^*)\sin^2\alpha\} = 0. \tag{2.37}$$

Further, from arguments identical to those in § 2 *d*, we have

$$\sin\phi_2 = \zeta_2^*(1 - \tfrac{1}{4}\zeta_2^{*^2})^{1/2}, \qquad \cos\phi_2 = 1 - \tfrac{1}{2}\zeta_2^{*^2}. \tag{2.38}$$

Equations (2.12) and (2.36)–(2.38) enable us to determine ζ_1^*, ζ_2^* for given values of β and W/S. The horizontal force H between the two strips is now given by equation (2.13), whence

$$H = \tfrac{1}{2}S\zeta_1^{*^2}, \tag{2.39}$$

and if this is expressed in dimensional form, namely

$$H = \tfrac{1}{2}z_1^{*^2}\gamma g, \tag{2.40}$$

it identifies H as the resultant of the horizontal component of hydrostatic pressures acting over the range $0 < z < z_1^*$ in regions remote from the strips.

For the linear theory, equations (2.38) yield

$$\phi_2 = \zeta_2^*, \tag{2.41}$$

and equations (2.17) may be solved to yield

$$\zeta_1^* = \frac{W}{S}\left(\frac{3 + 3\beta + \beta^2}{3 + 6\beta + 4\beta^2 + 2\beta^3}\right), \qquad \zeta_2^* = \frac{W}{S}\left(\frac{3 + \beta^2}{3 + 6\beta + 4\beta^2 + 2\beta^3}\right). \tag{2.42}$$

The variation of H with β is now given by equation (2.18), but it is best expressed

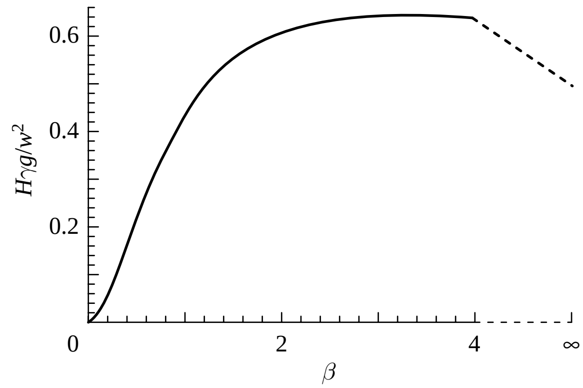

Figure 4. Variation of force of mutual attraction H with width of touching, shallow strips.

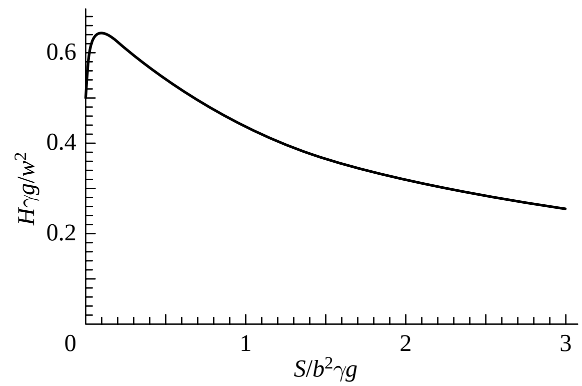

Figure 5. Variation of force of mutual attraction H with surface tension S for touching, shallow strips of given width.

in terms of w, the weight per unit *area*, so that $W = 2bw$. Thus we find

$$H = \frac{2w^2}{\gamma g}\left(\frac{\beta(3+3\beta+\beta^2)}{3+6\beta+4\beta^2+2\beta^3}\right)^2, \tag{2.43}$$

which is shown in figure 4.

The corresponding variation of H with surface tension S for strips of given width must take account of the fact that β varies with S because of the variation of L with S. Thus if we introduce

$$q = \beta^{-2} = \frac{S}{b^2\gamma g}, \tag{2.44}$$

we can express equation (2.43) in the form

$$H = \frac{2w^2}{\gamma g}\left(\frac{1+3q^{1/2}+3q}{2+4q^{1/2}+6q+3q^{3/2}}\right)^2, \tag{2.45}$$

which is shown in figure 5. It is seen that H increases with S when $S < 0.0935b^2\gamma g$; elsewhere an increase in S results in a decrease in H. Indeed, as $b \to 0$, so that the proportion of support from hydrostatic pressure is negligible, H varies *inversely* as S.

The variation of $H_{\text{exact}}/H_{\text{linear}}$ with $W/(1+\beta)S$ is shown in figure 6 for various values of β. Also shown are curves along which $\phi_2 = 25$, 30 and 35°. It is seen that

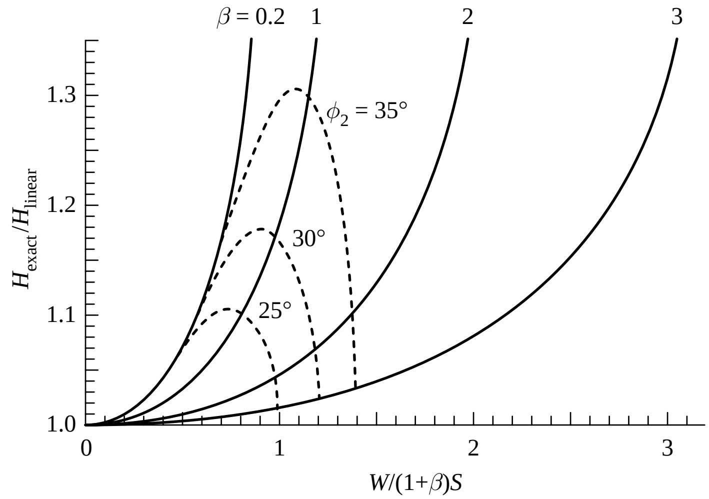

Figure 6. Variation of $H_{\text{exact}}/H_{\text{linear}}$ with $W/(1+\beta)S$.

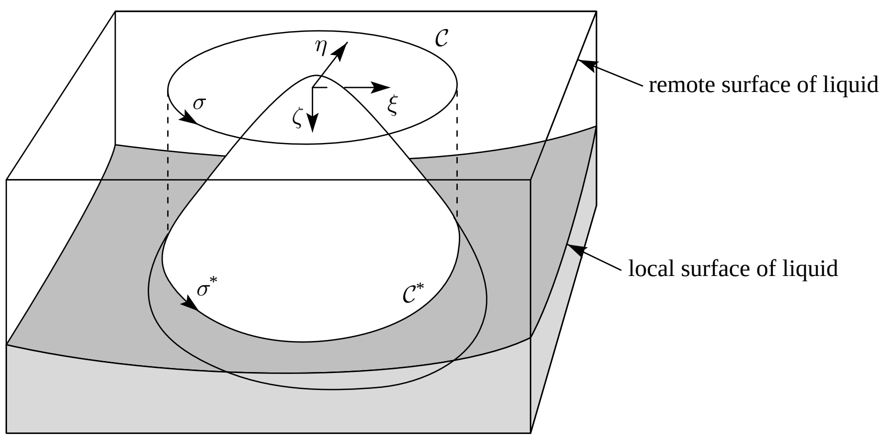

Figure 7. Notation for arbitrary object showing coordinate system and closed curves $\mathcal{C}, \mathcal{C}^*$.

if the errors given by the linear theory are not to exceed about 10%, the angle ϕ_2 should not exceed about 25°.

3. Equilibrium conditions for object of arbitrary shape

The object of arbitrary shape is depicted in figure 7. The surface of the liquid meets the object around a closed curve $\mathcal{C}^*$ where at a given point σ^* the deflexion is denoted by ζ^* and the slope of the surface relative to a horizontal normal to $\mathcal{C}^*$ is denoted by ϕ. The axes $O\xi$, $O\eta$ are in the horizontal plane asymptotic to the remote surface of the liquid.

We have already introduced non-dimensional measures of lengths by expressing them in units of L, and it is now convenient to introduce the following:

$$\begin{aligned} \text{weights per unit volume in units of } & \gamma g, \\ \text{pressures in units of } & L\gamma g, \\ \text{loads per unit length in units of } S \text{ or } & L^2\gamma g, \end{aligned}$$

$$\text{forces in units of } SL \text{ or } L^3\gamma g,$$
$$\text{moments in units of } SL^2 \text{ or } L^4\gamma g.$$

Note, for example, that because γg is defined as the unit value of weight per unit volume, the non-dimensional value of the weight of a given volume of liquid is equal to the non-dimensional value of its volume. The alternative units are identical because of the definition of L, but each has greater relevance when purely surface tension or hydrostatic pressure is involved.

The object is supported by surface tension around $\mathcal{C}^*$ and by hydrostatic pressure below $\mathcal{C}^*$, so that in a formal sense we can write

$$\mathcal{R}(\text{object}) = \mathcal{R}_{\text{st}}(\text{object}) + \mathcal{R}_{\text{hp}}(\text{object}), \tag{3.1}$$

where $\mathcal{R}$ denotes 'resultant load applied to' and suffices indicate by which mode of support. We consider first the resultant forces and moments caused by surface tension.

(*a*) *Resultants of surface tension forces acting on object*

Here we are concerned with certain contour integrals around $\mathcal{C}^*$, a curve in three dimensions with coordinates (ξ^*, η^*, ζ^*). However, these can be reduced to contour integrals around a curve in purely (ξ^*, η^*) coordinates. We denote by $\mathcal{C}$, σ the vertical projection of $\mathcal{C}^*, \sigma^*$ onto the horizontal plane where $\zeta = 0$, so that

$$(\mathrm{d}\sigma^*)^2 = (\mathrm{d}\sigma)^2 + (\mathrm{d}\zeta^*)^2. \tag{3.2}$$

It follows that

$$\frac{\mathrm{d}\sigma^*}{\mathrm{d}\sigma} = 1 + \tfrac{1}{2}\left(\frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma}\right)^2 + \mathcal{O}\left\{\left(\frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma}\right)^4\right\}, \tag{3.3}$$

and we note that within the framework of linear theory,

$$\left(\frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma}\right)^2 \ll 1. \tag{3.4}$$

Equation (3.3) is required only in the determination of the resultant horizontal forces due to surface tension; elsewhere, it is appropriate in linear theory to combine equations (3.3), (3.4) to yield the simpler result that

$$\frac{\mathrm{d}\sigma^*}{\mathrm{d}\sigma} = 1. \tag{3.5}$$

(i) *Resultant vertical force due to surface tension*

The surface tension acting over a length $\delta\sigma^*$ of the curve $\mathcal{C}^*$ has a vertical component $\phi\delta\sigma^*$ and hence the resultant vertical force due to surface tension is given by

$$\oint \phi\,\mathrm{d}\sigma^* \approx \oint \phi\,\mathrm{d}\sigma, \tag{3.6}$$

where the integrals are taken around $\mathcal{C}^*$ and $\mathcal{C}$, respectively.

(ii) *Resultant moments due to surface tension*

In like manner, the vertical component $\phi\delta\sigma^*$ of the surface tension causes resultant clockwise moments about the axes $O\xi$, $O\eta$ equal to

$$\oint \eta^* \phi \, \mathrm{d}\sigma \quad \text{and} \quad -\oint \xi^* \phi \, \mathrm{d}\sigma. \tag{3.7}$$

Now for small values of ϕ, $\cos\phi \approx 1$ and hence the horizontal component directed along ν is $\delta\sigma^*$. This can be resolved into components $\delta\sigma^* \cos\Theta$ and $\delta\sigma^* \sin\Theta$ in the ξ and η directions, where Θ is the angle between ν and $O\xi$. These components therefore result in the following moments about the axes $O\xi$, $O\eta$:

$$\oint \zeta^* \sin\Theta \, \mathrm{d}\sigma \quad \text{and} \quad -\oint \zeta^* \cos\Theta \, \mathrm{d}\sigma. \tag{3.8}$$

(iii) *Resultant horizontal forces due to surface tension*

In determining the resultant horizontal forces due to surface tension we note that *constant*, horizontal, planar loads per unit length directed normal to the boundary of an arbitrarily shaped object have no resultant horizontal force nor, indeed, a resultant torque about the axis $O\zeta$. Such forces therefore arise only from loads per unit length that deviate from a constant value; in other words, the horizontal component per unit length due to surface tension must be given its exact value, namely $\cos\phi$, although in linear theory this is treated as $(1 - \frac{1}{2}\phi^2)$. Thus we have seen that for an infinite strip the horizontal force due to surface tension is of order ϕ^2, but this simple example is characterised by the constancy of ϕ and, more importantly, the vanishing of the slope $\mathrm{d}\zeta^*/\mathrm{d}\sigma^*$ along the (infinite) boundaries. When $\mathrm{d}\zeta^*/\mathrm{d}\sigma^* \neq 0$, account must be taken of this feature because it corresponds to a rotation of the local surface about the horizontal vector ν. If $\phi = 0$, the direction of the surface tension normal to $\delta\sigma^*$ would be along ν, but if $\phi \neq 0$, this direction is inclined at an angle $\phi \mathrm{d}\zeta^*/\mathrm{d}\sigma$ to the vector ν. It follows that the resultant force in the ξ direction due to surface tension is given by

$$\oint \cos\phi \cos\left(\Theta + \phi \frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma}\right) \mathrm{d}\sigma^* = -\oint \left[\frac{1}{2}\left\{\phi^2 - \left(\frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma}\right)^2\right\} \cos\Theta + \phi \frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma} \sin\Theta\right] \mathrm{d}\sigma, \tag{3.9}$$

within the context of linear theory. Likewise, the resultant force in the η direction is given by

$$\oint \cos\phi \sin\left(\Theta + \phi \frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma}\right) \mathrm{d}\sigma^* = -\oint \left[\frac{1}{2}\left\{\phi^2 - \left(\frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma}\right)^2\right\} \sin\Theta - \phi \frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma} \cos\Theta\right] \mathrm{d}\sigma. \tag{3.10}$$

These forces are regarded as acting along the axes $O\xi$, $O\eta$ so that they do not contribute to a torque about the axis $O\zeta$.

(iv) *Resultant torque due to surface tension*

The resultant torque about the axis $O\zeta$ stems from the horizontal components $\delta\sigma^* \cos\phi$ that are inclined at an angle $\phi \, \mathrm{d}\zeta^*/\mathrm{d}\sigma$ to the vector ν, as discussed above.

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The length of the perpendicular from $O\zeta$ to the line of action of this elemental load is

$$\rho^* \sin\left(\theta - \Theta - \phi\frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma}\right), \tag{3.11}$$

where ρ^*, θ are the polar coordinates of a point on $\mathcal{C}$. It follows that the resultant clockwise torque due to surface tension is given by

$$\oint \rho^* \sin\left(\theta - \Theta - \phi\frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma}\right)\cos\phi\,\mathrm{d}\sigma^*$$
$$= -\oint \rho^*\left[\frac{1}{2}\left\{\phi^2 - \left(\frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma}\right)^2\right\}\sin(\theta - \Theta) + \phi\frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma}\cos(\theta-\Theta)\right]\mathrm{d}\sigma, \tag{3.12}$$

within the framework of linear theory.

(*b*) *Resultants of hydrostatic pressure acting on object*

To determine the resultants of the hydrostatic pressure it is convenient first to consider the equilibrium of a related volume of liquid. Let $\mathcal{V}$ (in units of L^3) be the volume whose boundaries comprise the horizontal plane bounded by $\mathcal{C}$, the vertical cylindrical region between $\mathcal{C}$ and $\mathcal{C}^*$, and the wetted part of the object below $\mathcal{C}^*$. Consider now the hydrostatic pressure applied to an identically shaped volume $\mathcal{V}$ of liquid in regions remote from any surface irregularity. This volume $\mathcal{V}$ is necessarily in equilibrium with the surrounding liquid, and the resultant of the hydrostatic pressure applied to it is solely a vertical force $\mathcal{V}$ (in units of $L^3\gamma g$) directed through the centre of gravity of the volume $\mathcal{V}$. Now the hydrostatic pressures applied to the object are confined to regions below $\mathcal{C}^*$, where they are identical to those applied to the liquid volume $\mathcal{V}$. Of course, the liquid volume $\mathcal{V}$ also has hydrostatic pressures applied to the cylindrical region between $\mathcal{C}$ and $\mathcal{C}^*$ so that, in the formal notation of equation (3.1), we can write

$$\mathcal{R}_{\mathrm{hp}}(\text{volume } \mathcal{V}) = \mathcal{R}_{\mathrm{hp}}(\text{object}) + \mathcal{R}_{\mathrm{hp}}(\text{cylinder } \mathcal{C}\mathcal{C}^*). \tag{3.13}$$

Equation (3.13) enables us to express $\mathcal{R}_{\mathrm{hp}}(\text{object})$ in terms of $\mathcal{R}_{\mathrm{hp}}(\text{volume } \mathcal{V})$, which is known, and the values of $\mathcal{R}_{\mathrm{hp}}(\text{cylinder } \mathcal{C}\mathcal{C}^*)$ which we now determine.

(i) *Resultant vertical force on object due to hydrostatic pressure*

The hydrostatic pressures applied to the cylinder $\mathcal{C}\mathcal{C}^*$ are directed normal to the cylindrical surface and accordingly have zero vertical resultant. Thus we have the simple result that the resultant vertical force on the object due to hydrostatic pressure is equal to $\mathcal{V}$.

(ii) *Resultant horizontal forces on object due to hydrostatic pressure*

Because the liquid volume $\mathcal{V}$ is in equilibrium under its own gravitational weight, the hydrostatic pressures applied to it have no resultant horizontal force. Hence, from equation (3.13), the resultant horizontal forces on the object due to hydrostatic pressure are of the same magnitude but opposite sign to those applied to the cylinder $\mathcal{C}\mathcal{C}^*$ where, along any vertical strip bounded by generators at σ and $\sigma + \delta\sigma$, the resultant of the hydrostatic pressure acting in the direction of $-\nu$, is given by

$$\delta\sigma \int_0^{\zeta^*} \zeta\,\mathrm{d}\zeta. \tag{3.14}$$

It follows that the resultants of the hydrostatic pressures acting on the object in the ξ and η directions are, respectively,

$$\frac{1}{2}\oint \zeta^{*^2}\cos\Theta\,\mathrm{d}\sigma \quad \text{and} \quad \frac{1}{2}\oint \zeta^{*^2}\sin\Theta\,\mathrm{d}\sigma. \tag{3.15}$$

These forces are regarded as acting along the axes $O\xi$, $O\eta$ so that they do not contribute to a torque about the axis $O\zeta$.

(iii) *Resultant moments on object due to hydrostatic pressure*

As noted previously, the liquid volume $\mathcal{V}$ is in equilibrium with a vertical force $\mathcal{V}$ that passes through the centre of gravity of $\mathcal{V}$ with coordinates $(\xi_{\mathrm{cg},\mathcal{V}}, \eta_{\mathrm{cg},\mathcal{V}}, \zeta_{\mathrm{cg},\mathcal{V}})$, say. It follows that there are clockwise moments about the axes $O\xi$, $O\eta$ applied by hydrostatic pressure to the liquid volume equal to

$$\mathcal{V}\eta_{\mathrm{cg},\mathcal{V}} \quad \text{and} \quad -\mathcal{V}\xi_{\mathrm{cg},\mathcal{V}}. \tag{3.16}$$

However, from equation (3.1) we should also consider the moments due to hydrostatic pressure acting over the cylinder $\mathcal{CC}^*$. From arguments similar to those in the previous section, these moments are respectively

$$-\frac{1}{3}\oint \zeta^{*^3}\sin\Theta\,\mathrm{d}\sigma \quad \text{and} \quad \frac{1}{3}\oint \zeta^{*^3}\cos\Theta\,\mathrm{d}\sigma, \tag{3.17}$$

but comparisons with the corresponding moments due to surface tension, see expressions (3.7) and (3.8), show that the terms in (3.17) can be ignored in linear theory.

(iv) *Resultant torque applied to object by hydrostatic pressure*

From arguments similar to those used in determining the horizontal forces, the resultant torque applied to the object by hydrostatic pressure is of equal magnitude but opposite sign to that applied to the cylinder $\mathcal{CC}^*$, where the hydrostatic pressure acting over a vertical strip of width $\delta\sigma$ yields a force $\frac{1}{2}\zeta^{*^2}\delta\sigma$ directed along $-\nu$. Now the length of the perpendicular from $O\zeta$ to the line of action of this elemental load is $\rho^*\sin(\theta-\Theta)$, and hence the clockwise torque about $O\zeta$ applied to the object by hydrostatic pressure is

$$\frac{1}{2}\oint \rho^*\zeta^{*^2}\sin(\theta-\Theta)\,\mathrm{d}\sigma. \tag{3.18}$$

(c) *Equilibrium equations for object*

It was shown in § 2 that the boundary conditions for the adjacent liquid are provided by the equations of vertical equilibrium of the strip, and moment equilibrium about the horizontal axis $O\eta$. The force of mutual attraction was then determined by the derived equilibrium configuration. A similar situation occurs for an object of arbitrary shape where the boundary conditions for the surrounding liquid are provided by the equations of vertical equilibrium of the object and moment equilibrium about the axes $O\xi$, $O\eta$. The horizontal forces and the torque about the axis $O\zeta$ are determined by the derived equilibrium configuration. Now from equation (3.1) and § 3 *a*, *b* we can write down the resultant forces and moments applied to the object by surface tension and hydrostatic pressure, and we can equilibrate these against the gravitational forces on the object. Hence the equation of vertical equilibrium is given by

$$\oint \phi \, \mathrm{d}\sigma + \mathcal{V} = \mathcal{W}, \tag{3.19}$$

where $\mathcal{W}$ is the weight of the object expressed in units of $L^3\gamma g$.

Likewise, if the (ξ, η) coordinates of the centre of gravity of the object in its equilibrium state are $(\xi_{\mathrm{cg},\mathcal{W}},\ \eta_{\mathrm{cg},\mathcal{W}})$, the equations of moment equilibrium are given by

$$\oint (\eta^* \phi + \zeta^* \sin \Theta) \, \mathrm{d}\sigma + \mathcal{V}\eta_{\mathrm{cg},\mathcal{V}} = \mathcal{W}\eta_{\mathrm{cg},\mathcal{W}}, \tag{3.20}$$

and

$$\oint (\xi^* \phi + \zeta^* \cos \Theta) \, \mathrm{d}\sigma + \mathcal{V}\xi_{\mathrm{cg},\mathcal{V}} = \mathcal{W}\xi_{\mathrm{cg},\mathcal{W}}. \tag{3.21}$$

Note that for reasons similar to those discussed in § 2 *c*, the above equations do not include any contribution stemming from the forces of mutual attraction. In Appendix A it is shown that the equation of vertical equilibrium, (3.19) is consistent with the principle of Archimedes *ca.* 250 BC (see Heath 1897).

(i) *Forces of mutual attraction*

Equations (3.19)–(3.21) provide the boundary conditions for the surrounding liquid to enable $\mathcal{C}$ and the values of ζ^* and ϕ on $\mathcal{C}^*$ to be determined. The horizontal forces in the ξ and η directions, namely $\mathcal{H}_\xi$ and $\mathcal{H}_\eta$, that are required to maintain the static equilibrium of the object are now given by equations (3.9), (3.10) and (3.15):

$$\mathcal{H}_\xi = \oint \left[\frac{1}{2} \left\{ \phi^2 - {\zeta^*}^2 - \left(\frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma} \right)^2 \right\} \cos \Theta + \phi \frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma} \sin \Theta \right] \mathrm{d}\sigma, \tag{3.22}$$

and

$$\mathcal{H}_\eta = \oint \left[\frac{1}{2} \left\{ \phi^2 - {\zeta^*}^2 - \left(\frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma} \right)^2 \right\} \sin \Theta - \phi \frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma} \cos \Theta \right] \mathrm{d}\sigma. \tag{3.23}$$

These forces are regarded as acting along the axes $O\xi$ and $O\eta$; the actual line of action of their resultant $(\mathcal{H}_\xi^2 + \mathcal{H}_\eta^2)^{1/2}$ in the ξ, η plane is determined by the clockwise torque $\mathcal{T}_\zeta$ about the axis $O\zeta$ required for equilibrium where, from (3.12) and (3.18),

$$\mathcal{T}_\zeta = \oint \rho^* \left[\frac{1}{2} \left\{ \phi^2 - {\zeta^*}^2 - \left(\frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma} \right)^2 \right\} \sin(\theta - \Theta) + \phi \frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma} \cos(\theta - \Theta) \right] \mathrm{d}\sigma. \tag{3.24}$$

(ii) *Complex variable representation*

The forces $\mathcal{H}_\xi$, $\mathcal{H}_\eta$ may be expressed compactly in the form,

$$\mathcal{H}_\xi + \mathrm{i}\mathcal{H}_\eta = \frac{1}{2} \oint \left\{ \left(\phi - \mathrm{i}\frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma} \right)^2 - {\zeta^*}^2 \right\} \mathrm{e}^{\mathrm{i}\Theta} \, \mathrm{d}\sigma. \tag{3.25}$$

(*d*) *Transfer of non-uniform, horizontal forces through the liquid*

Consider the (non-dimensional) tensile and shear forces per unit length, $\mathcal{N}_\xi$, $\mathcal{N}_\eta$, $\mathcal{N}_{\xi\eta}$ acting in a horizontal plane on an element of liquid surface plus underlying liquid bounded by $\delta\xi$, $\delta\eta$ and a plane where $\zeta = \zeta_D$, say, which is below any surface deflexion. The component of $\mathcal{N}_\xi$, say, due to hydrostatic pressure is given by

$$-\int_{\zeta}^{\zeta_D} \zeta \, \mathrm{d}\zeta = \tfrac{1}{2}(\zeta^2 - \zeta_D^2), \tag{3.26}$$

and that due to surface tension is

$$\left\{1 + \tfrac{1}{2}\left(\frac{\partial \zeta}{\partial \eta}\right)^2\right\}\left\{1 - \tfrac{1}{2}\left(\frac{\partial \zeta}{\partial \xi}\right)^2\right\}, \tag{3.27}$$

where the first term stems from the increased length caused by the slope $\partial\zeta/\partial\eta$ and the second term is the 'cosine' factor associated with the slope $\partial\zeta/\partial\xi$. Combining these two components within the context of linear theory gives

$$\mathcal{N}_\xi = \frac{1}{2}\left\{\zeta^2 + \left(\frac{\partial \zeta}{\partial \eta}\right)^2 - \left(\frac{\partial \zeta}{\partial \xi}\right)^2\right\}, \tag{3.28}$$

where we have omitted the term $(1 - \frac{1}{2}\zeta_D^2)$ because it is a component of a self-equilibrating, constant, all-round tension. Likewise we have

$$\mathcal{N}_\eta = \frac{1}{2}\left\{\zeta^2 + \left(\frac{\partial \zeta}{\partial \xi}\right)^2 - \left(\frac{\partial \zeta}{\partial \eta}\right)^2\right\}. \tag{3.29}$$

Hydrostatic pressure does not contribute to the shearing force which stems solely from a surface tension resultant dependent on the surface slopes:

$$\mathcal{N}_{\xi\eta} = -\frac{\partial \zeta}{\partial \xi}\frac{\partial \zeta}{\partial \eta}. \tag{3.30}$$

For equilibrium we require

$$\frac{\partial \mathcal{N}_\xi}{\partial \xi} + \frac{\partial \mathcal{N}_{\xi\eta}}{\partial \eta} = 0, \qquad \frac{\partial \mathcal{N}_\eta}{\partial \eta} + \frac{\partial \mathcal{N}_{\xi\eta}}{\partial \xi} = 0, \tag{3.31}$$

whence, from equations (3.28)–(3.30),

$$\frac{\partial \zeta}{\partial \xi}(\zeta - \nabla^2\zeta) = 0, \qquad \frac{\partial \zeta}{\partial \eta}(\zeta - \nabla^2\zeta) = 0. \tag{3.32}$$

The above equations are therefore satisfied because the surface deflexion satisfies equation (2.15) which was derived from vertical equilibrium. In fact, one could argue that Laplace's surface tension equation could equally well have been deduced from considerations of horizontal equilibrium!

If we now express ϕ and $\mathrm{d}\zeta^*/\mathrm{d}\sigma$ in terms of $\partial\zeta/\partial\xi$ and $\partial\zeta/\partial\eta$, equations (3.22)–(3.24) become

$$\left.\begin{aligned}
\mathcal{H}_\xi &= -\oint (\mathcal{N}_\xi \cos\Theta + \mathcal{N}_{\xi\eta}\sin\Theta)\,\mathrm{d}\sigma,\\
\mathcal{H}_\eta &= -\oint (\mathcal{N}_\eta \sin\Theta + \mathcal{N}_{\xi\eta}\cos\Theta)\,\mathrm{d}\sigma,\\
\mathcal{T}_\zeta &= \oint \{\xi(\mathcal{N}_\eta \sin\Theta + \mathcal{N}_{\xi\eta}\cos\Theta) - \eta(\mathcal{N}_\xi \cos\Theta + \mathcal{N}_{\xi\eta}\sin\Theta)\}\,\mathrm{d}\sigma.
\end{aligned}\right\} \tag{3.33}$$

Further, because the horizontal forces $\mathcal{N}$ are in equilibrium throughout the free surface region of the liquid, the contour integrals of equations (3.22)–(3.24) need not

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necessarily follow the curve $\mathcal{C}$, but may include arbitrary regions of adjacent liquid surface whose outer boundary would be chosen to simplify the evaluation of the contour integrals.

(e) *Simplified expression for $\mathcal{H}$ between mirror-image objects*

A simple application of the above feature is in the determination of $\mathcal{H}$ between two mirror-image objects facing each other on an infinite expanse of liquid. If the objects are symmetrically disposed about the η-axis, say, we extend the path of the contour integral for $\mathcal{H}_\xi$ to include the semi-infinite expanse where ξ is positive. Because ζ is zero at infinity and, from symmetry, $\partial\zeta/\partial\xi$ is zero along the η-axis it follows that

$$\mathcal{H}_\xi = \frac{1}{2}\int_{-\infty}^{\infty}\left\{\zeta_{\mathrm{cl}}^2 + \left(\frac{\mathrm{d}\zeta_{\mathrm{cl}}}{\mathrm{d}\eta}\right)^2\right\}\mathrm{d}\eta, \tag{3.34}$$

where ζ_{cl} is the deflexion along the centre-line between the objects. Likewise, for an infinite array of identical paired mirror-image objects at a pitch p,

$$\mathcal{H}_\xi = \frac{1}{2}\int_{0}^{p}\left\{\zeta_{\mathrm{cl}}^2 + \left(\frac{\mathrm{d}\zeta_{\mathrm{cl}}}{\mathrm{d}\eta}\right)^2\right\}\mathrm{d}\eta. \tag{3.35}$$

(f) *Overall equilibrium of arbitrary number of objects*

For an arbitrary number of objects on an infinite expanse of liquid, integration of equations (3.32)–(3.34) around a contour of infinite radius yields the following equations of overall equilibrium:

$$\sum\mathcal{H}_\xi = \sum\mathcal{H}_\eta = \sum\mathcal{T}_\zeta = 0, \tag{3.36}$$

where the summations extend over each object.

(g) *The equilibrium state of strips that do not coalesce*

In § 4 we show that certain infinitely long strips do not coalesce but have an equilibrium state where they are separated. If the strips are aligned parallel to the η-axis, the load per unit length $\mathcal{N}_\xi$ is zero in the region between such strips and hence, from equation (3.28), the deflexion ζ satisfies the equation

$$\zeta^2 - \left(\frac{\mathrm{d}\zeta}{\mathrm{d}\xi}\right)^2 = 0, \tag{3.37}$$

whose solution is

$$\zeta = C\mathrm{e}^{\pm\xi}, \tag{3.38}$$

where C is a constant. Of course, the deflexion ζ must also satisfy equation (2.16) whose general solution is thus restricted to a single exponential variation. It follows that if ζ_1^*, ζ_2^* are the deflexions of the facing edges of two strips that are in equilibrium, the separation distance is such that

$$[\lambda]_{\mathrm{equilibrium}} = \left|\ln\left(\frac{\zeta_1^*}{\zeta_2^*}\right)\right|. \tag{3.39}$$

4. Phenomena associated with objects supported by surface tension

In order to demonstrate as simply as possible some of the phenomena associated with objects supported primarily by surface tension, in our first application of the general equilibrium equations (3.19)–(3.21) we again discuss the one-dimensional case of infinite cylindrical strips whose contact with the liquid is confined to their flat lower surfaces but, unlike the strips of shallow rectangular section discussed in § 2, the centre of gravity may be at an arbitrary point in the cross-section. The specific phenomena that we identify and analyse include: toppling instability, zero mutual attraction, localized and extensive mutual repulsion, stable and unstable states near a boundary, and characteristic deflexion patterns in rafts. We also consider infinite strips with pie-crust edge undulations that, prior to coalescence, can result in a phenomenon of mutual alignment, and a marked increase in the force of mutual attraction after coalescence.

To enable us to derive the general equilibrium equations for a single strip under arbitrary edge conditions we introduce additional orthogonal axes $O'\xi'$, $O'\zeta'$ that are embedded in the strip cross-section, where $O'\xi'$ lies on the flat lower surface of the strip whose edges are at $\xi' = \pm\beta$. The (ξ', ζ') coordinates of the centre of gravity of the strip are given by

$$\xi'_{\mathrm{cg},\mathcal{W}} = \epsilon\beta, \qquad \zeta'_{\mathrm{cg},\mathcal{W}} = -\tau\beta, \tag{4.1}$$

so that ϵ is a measure of the eccentricity of the centre of gravity and, for example, $\tau = h/b$ for strips of rectangular cross-section and uniform density. It follows that in a tilted state,

$$\xi_{\mathrm{cg},\mathcal{W}} = \epsilon\beta - \tfrac{1}{2}\tau(\zeta_1^* - \zeta_2^*). \tag{4.2}$$

Referring now to the general equilibrium equations (3.19), (3.21), we note that per unit length of strip:

$$\mathcal{W} = \frac{W}{S}, \qquad \mathcal{V} = \beta(\zeta_1^* + \zeta_2^*), \qquad \xi_{\mathrm{cg},\mathcal{V}} = -\tfrac{1}{3}\beta\left(\frac{\zeta_1^* - \zeta_2^*}{\zeta_1^* + \zeta_2^*}\right), \tag{4.3}$$

and hence the equations of vertical and moment equilibrium are

$$\left.\begin{aligned} \phi_1 + \phi_2 + \beta(\zeta_1^* + \zeta_2^*) &= W/S, \\ \beta(\phi_1 - \phi_2) + (1 + \tfrac{1}{3}\beta^2)(\zeta_1^* - \zeta_2^*) &= \{\tfrac{1}{2}\tau(\zeta_1^* - \zeta_2^*) - \epsilon\beta\}W/S. \end{aligned}\right\} \tag{4.4}$$

(a) Equilibrium of a single strip with arbitrary centre of gravity

In regions remote from a single strip the deflexion of the liquid surface tends to zero. Thus from equation (2.16) the deflexion of the liquid surface on either side of the strip is given by

$$\zeta_1 = \zeta_1^* \mathrm{e}^{-\nu_1}, \qquad \zeta_2 = \zeta_2^* \mathrm{e}^{-\nu_2}, \tag{4.5}$$

where ν_1 and ν_2 are measured normal to the edges of the strip. It follows that

$$\phi_1 = \zeta_1^*, \qquad \phi_2 = \zeta_2^*, \tag{4.6}$$

and the equilibrium equations (4.4) may be solved to yield

$$\zeta_1^*,\ \zeta_2^* = \frac{W}{2S}\left(\frac{1}{1+\beta} \mp \frac{\epsilon\beta}{1 + \beta + \frac{1}{3}\beta^2 - \frac{1}{2}\tau W/S}\right). \tag{4.7}$$

(i) Requirements for the assumption of shallowness

From equation (4.7) we see that the assumption of shallowness made in § 2 is valid if

$$\tau \ll 2S(1 + \beta + \tfrac{1}{3}\beta^2)/W. \tag{4.8}$$

Now, from equations (2.25) and the discussion at the end of § 2 *d*, the maximum value of W for which linear theory is satisfactory is approximately $\sqrt{2S(1+\beta)}$ and hence, even at this high value of W, the assumption of shallowness is valid if

$$\tau \ll \sqrt{2}\left(1+\frac{\beta^2}{3(1+\beta)}\right). \tag{4.9}$$

(ii) *Toppling instability of strip*

From equation (4.7) it is seen that as $\epsilon \to 0$ and

$$\tau W \to 2S(1+\beta+\tfrac{1}{3}\beta^2), \tag{4.10}$$

the expressions for ζ_1^*, ζ_2^* become indeterminate, which is indicative of a toppling instability. Note that the term on the left of equation (4.10) equals the moment caused by the displacement of the centre of gravity per angle of tilt, while the term on the right equals the restoring moment due to surface tension and hydrostatic pressure.

(iii) *Strips with zero mutual attraction*

Referring again to equation (4.7) and focusing attention on positive values of ϵ, it follows that when

$$\begin{aligned}\epsilon &= \frac{1}{\beta}\left(1+\frac{2\beta^2-3\tau W/S}{6(1+\beta)}\right),\\ &= \epsilon^*, \qquad \text{say},\end{aligned} \tag{4.11}$$

the deflexion ζ_1^* is zero, so that the surface of the liquid is undisturbed away from the lighter side of the strip. Two such mirror-image strips with their lighter sides facing each other therefore exhibit no force of mutual attraction. When $\epsilon > \epsilon^*$, the deflexion ζ_1^* is negative and hence the lighter edge of the strip is tilted above the remote level of the liquid.

(*b*) *Mutual attraction between two mirror-image strips*

Here we determine the force of mutual attraction between two arbitrary mirror-image strips and its dependence on β, ϵ, τ and λ, the (non-dimensional) distance between them. Because of symmetry we can confine attention to the right-hand strip, say, where from equation (2.16) the surface deflexion between the strips is given by

$$\zeta_1 = \zeta_1^* \cosh(\tfrac{1}{2}\lambda - \nu_1)\,\mathrm{sech}\,\tfrac{1}{2}\lambda, \tag{4.12}$$

where ν_1 is measured from edge$_1$, and the deflexion away from edge$_2$ is given by

$$\zeta_2 = \zeta_2^* \mathrm{e}^{-\nu_2}. \tag{4.13}$$

It follows that

$$\phi_1 = \zeta_1^* \tanh \tfrac{1}{2}\lambda, \qquad \phi_2 = \zeta_2^*, \tag{4.14}$$

and equations (4.4), (4.14) enable us to determine ζ_1^* and ζ_2^*, while from equations (3.22) or (3.34) the force of mutual attraction per unit length is given by

$$\begin{aligned}H &= \tfrac{1}{2}S{\zeta_1^*}^2\,\mathrm{sech}^2\,\tfrac{1}{2}\lambda,\\ &= \tfrac{1}{2}S\zeta_{\mathrm{cl}}^2,\end{aligned} \tag{4.15}$$

where ζ_{cl} is the deflexion midway between the strips.

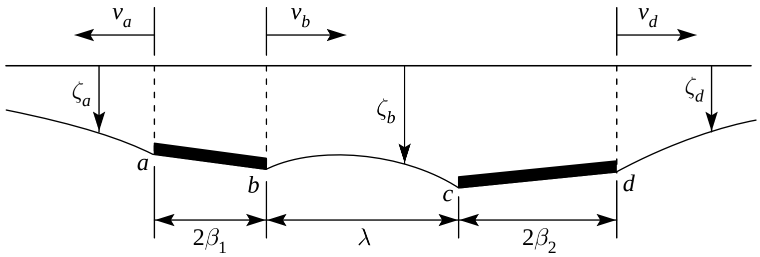

Figure 8. Notation for two unequal strips.

For arbitrary values of ϵ, the force of mutual attraction is most conveniently expressed in terms of that when ϵ is zero. Thus we find

$$H_{\epsilon=0} = \frac{W^2}{2S} e^{-\lambda} \left(\frac{J}{J(1+\beta) + \beta G e^{-\lambda}} \right)^2, \tag{4.16}$$

where

$$J = 3(1+\beta) + G,$$
$$G = \beta^2 - \tfrac{3}{2}\tau W/S.$$

For non-zero values of ϵ, H is given simply by

$$H = (1 - \epsilon/\epsilon^*)^2 H_{\epsilon=0}, \tag{4.17}$$

and we note that although H is zero if $\epsilon = \epsilon^*$, it is otherwise finite and positive. Note, too, that if $\epsilon = -\epsilon^*$, so that the strips have their heavier sides facing each other,

$$H = 4H_{\epsilon=0}. \tag{4.18}$$

(c) *Localized mutual repulsion*

The phenomenon of localized mutual repulsion is most simply displayed by consideration of the equilibrium of two shallow strips of rectangular cross-section for which ϵ is zero. However, unlike those considered in § 2 f, the strips have different weights W_1, W_2 and widths $2\beta_1$, $2\beta_2$, as shown in figure 8 where successive edges are denoted by a, b, c, d, and (ν, ζ) coordinate systems whose origins are at a, b and d are identified by the corresponding suffices.

The deflexions ζ_a, ζ_b and ζ_d are of the form

$$\left.\begin{aligned} \zeta_a &= C_1 e^{-\nu_a}, \\ \zeta_b &= C_2 e^{-\nu_b} + C_3 e^{\nu_b}, \\ \zeta_d &= C_4 e^{-\nu_d}, \end{aligned}\right\} \tag{4.19}$$

where the C_n are constants. It follows that

$$\left.\begin{aligned} \zeta_a^* &= C_1, & \phi_a &= C_1, \\ \zeta_b^* &= C_2 + C_3, & \phi_b &= C_2 - C_3, \\ \zeta_c^* &= C_2 e^{-\lambda} + C_3 e^{\lambda}, & \phi_c &= -C_2 e^{-\lambda} + C_3 e^{\lambda}, \\ \zeta_d^* &= C_4, & \phi_d &= C_4. \end{aligned}\right\} \tag{4.20}$$

The constants C_n are to be determined from the equilibrium equations (4.4) for each strip, while the force of mutual attraction is given by equation (2.18). Hence we find

$$H = \frac{UV\mathrm{e}^{-\lambda}(W_1U - W_2\beta_1^3\mathrm{e}^{-\lambda})(W_2V - W_1\beta_2^3\mathrm{e}^{-\lambda})}{2S(1+\beta_1)(1+\beta_2)(UV - \beta_1^3\beta_2^3\mathrm{e}^{-2\lambda})^2}, \tag{4.21}$$

where

$$U = (1+\beta_2)(3+3\beta_1+\beta_1^2),$$
$$V = (1+\beta_1)(3+3\beta_2+\beta_2^2).$$

The symmetry of the terms in the expression for H is because $H_{12} = H_{21}$. For sufficiently large values of λ, equation (4.21) shows that

$$\frac{HS}{W_1W_2} \sim \frac{\mathrm{e}^{-\lambda}}{2(1+\beta_1)(1+\beta_2)}, \tag{4.22}$$

which is essentially positive, indicating that there is then mutual attraction, but this is not necessarily so everywhere.

(i) *Conditions for mutual repulsion, H negative*

An examination of equation (4.21) shows that a negative value for H can only occur if one, and only one, of the factors in parentheses in the numerator is negative. However, it may readily be shown that when either of these factors is negative, the other is necessarily positive. Further, because equation (4.21) is unaltered by an interchange of the suffices 1, 2, it is sufficient to confine attention to a negative first factor, say. Mutual repulsion therefore occurs if

$$\frac{W_2}{1+\beta_2} > \frac{W_1\mathrm{e}^{\lambda}(3+3\beta_1+\beta_1^2)}{\beta_1^3}. \tag{4.23}$$

If this inequality holds for some value of λ, the position of stable equilibrium occurs where H is zero, i.e. when the inequality becomes an equality, so that

$$[\lambda]_{\text{equilibrium}} = \ln\left(\frac{W_2\beta_1^3}{W_1(1+\beta_2)(3+3\beta_1+\beta_1^2)}\right). \tag{4.24}$$

Note that $[\lambda]_{\text{equilibrium}}$ may also be determined without first deriving the expression for H. This is because the analysis of § 3 *g* indicates that the constant C_2 is zero, and hence the equilibrium equations (4.4) are sufficient to determine C_1, C_3, C_4 and $[\lambda]_{\text{equilibrium}}$. See also equation (3.39).

The inequality (4.23) is quite general in that it determines if there is mutual repulsion at a given value of λ. The tighter condition that two strips will or will not coalesce depends on whether

$$W_2/W_1 < \text{or} > (1+\beta_2)(3+3\beta_1+\beta_1^2)/\beta_1^3, \tag{4.25}$$

i.e.

$$\frac{w_2}{w_1} < \text{or} > \left(\frac{1+\beta_2}{\beta_2}\right)\left(\frac{3+3\beta_1+\beta_1^2}{\beta_1^2}\right), \tag{4.26}$$

which shows at a glance that strips with the same weight per unit area will always coalesce whatever their respective widths. It also shows that our earlier choice of a negative first factor implies that we have defined strip$_2$ as having the greater weight per unit area.

(ii) *De-coupling of the strip deflexions*

For strips that have coalesced, each of the deflexions ζ_a^*, $\zeta_b^*(=\zeta_c^*)$, ζ_d^* is a function of β_1, β_2, W_1 and W_2. But for strips whose equilibrium state is one of separation, with strip$_2$ having the greater weight per unit area, the deflexions are de-coupled:

$$\left.\begin{aligned}\zeta_a^* &= \frac{W_1}{2S\beta_1^3}\left(3-3\beta_1+\beta_1^2\right),\\ \zeta_b^* &= \frac{W_1}{2S\beta_1^3}\left(3+3\beta_1+\beta_1^2\right),\\ \zeta_c^* = \zeta_d^* &= \frac{W_2}{2S}\left(\frac{1}{1+\beta_2}\right).\end{aligned}\right\} \tag{4.27}$$

Furthermore, although strip$_1$ is tilted, strip$_2$ is horizontal with a deflexion identical to that if strip$_1$ was not present, see equation (2.25). The corresponding analysis for the exact theory is given in Appendix B.

(iii) *Discussion*

If β_1 or β_2 are small, equation (4.26) shows that mutual repulsion can only occur if $w_2 \gg w_1$, but if $\beta_1 = \beta_2 = 5$, say, we find that mutual repulsion occurs if $w_2/w_1 > 2.07$; for example, if we take $w_2/w_1 = 3$, equation (4.24) then shows that $[\lambda]_{\text{equilibrium}} = 0.37$, so that the range of mutual repulsion is quite small. Note, however, that because of the nature of equation (4.24) an e-fold increase, for example, in the weight ratio W_2/W_1 would, in theory, increase $[\lambda]_{\text{equilibrium}}$ by a unit amount, i.e. to 1.37. In practice, however, such high values of W_2/W_1 may not be adequately treated by the present analysis because an increase in W_2 increases its nonlinear behaviour, while a reduction in W_1 makes the assumption of its rigidity less tenable.

An understanding of why localized mutual repulsion can occur follows from a consideration of the limiting case in which $w_1 \to 0$ and $\beta_1 \to \infty$. Because we have assumed that this 'strip' is rigid, it simply acts as a device for maintaining the liquid level under it. To the other strip, this acts as a boundary where the liquid level is fixed. The deflexion of the strip in the immediate vicinity of this boundary will necessarily result in components of hydrostatic pressure and surface tension, each of which acts as a repulsion. Indeed, substitution of these limiting values into equation (4.24) shows that $[\lambda]_{\text{equilibrium}} \to \infty$, so that there is extensive mutual repulsion. To investigate such effects in a more realistic context we next consider a strip near a fixed boundary.

(*d*) *Equilibrium of strip near boundary*

In what follows we determine the equilibrium state of a shallow, rigid strip of weight per unit length W and width 2β which is at a distance λ from a fixed boundary where either the surface deflexion ζ_{B} or the surface slope ϕ_{B}, as defined in (2.19), is specified. For the analyses the notation of § 4 *c* was adopted insofar as it relates to the edges *a*, *b* of the strip, while the boundary takes the place of the edge *c*. First, however, we introduce

$$\zeta_{\text{hp}} = \frac{W}{2\beta S} = \frac{w}{\gamma g L}, \tag{4.28}$$

which is the (non-dimensional) surface deflexion due to hydrostatic pressure when surface tension plays a negligible role, e.g. when $\beta \to \infty$.

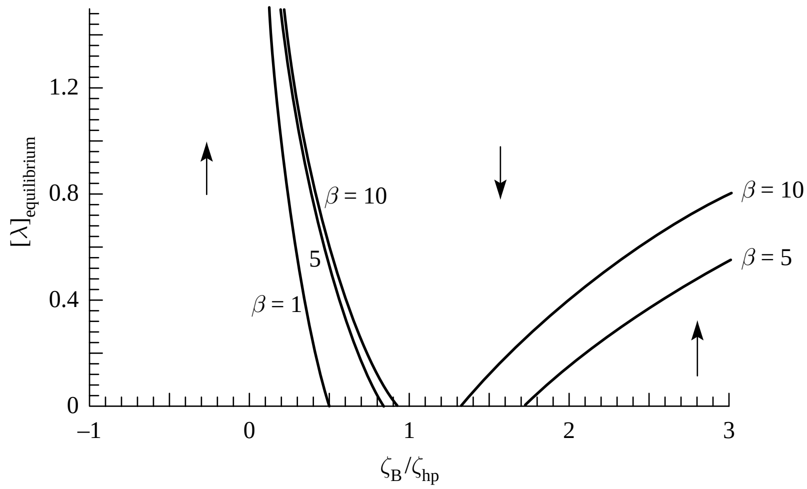

Figure 9. Variation of equilibrium gap with deflexion of liquid at boundary. Arrows indicate direction of force applied to strip away from equilibrium state. β is the semi-width of strip; λ is the distance from boundary.

(i) *Specified surface deflexion at boundary*

The constants C_1, C_2, C_3 are readily determined from the equilibrium equations (2.17) and hence from equation (2.18) we find

$$\frac{H}{2S} = \frac{\beta(1+\beta+\frac{1}{3}\beta^2)\mathrm{e}^{-\lambda}E_1E_2}{\{(1+\beta)(1+\beta+\frac{1}{3}\beta^2)-\frac{1}{3}\beta^3\mathrm{e}^{-2\lambda}\}^2}, \tag{4.29}$$

where

$$E_1 = \zeta_{\mathrm{B}}(1+\beta) - \zeta_{\mathrm{hp}}\beta\mathrm{e}^{-\lambda},$$
$$E_2 = \zeta_{\mathrm{hp}}(1+\beta+\tfrac{1}{3}\beta^2) - \tfrac{1}{3}\zeta_{\mathrm{B}}\beta^2\mathrm{e}^{-\lambda}.$$

Equation (4.29) shows that when ζ_{B} is zero or negative the term E_1 is negative and E_2 is positive for all values of λ, so that there is extensive mutual repulsion. For positive values of ζ_{B} the situation is more complex; thus, when

$$\frac{\beta}{1+\beta} < \frac{\zeta_{\mathrm{B}}}{\zeta_{\mathrm{hp}}} < \frac{3+3\beta+\beta^3}{\beta^2}, \tag{4.30}$$

there is mutual attraction for all values of λ, so that $[\lambda]_{\text{equilibrium}} = 0$. When $\zeta_{\mathrm{B}}/\zeta_{\mathrm{hp}}$ is below the range shown in (4.30) there is localized mutual repulsion and

$$[\lambda]_{\text{equilibrium}} = \ln\left(\frac{\beta\zeta_{\mathrm{hp}}}{(1+\beta)\zeta_{\mathrm{B}}}\right). \tag{4.31}$$

Likewise, when $\zeta_{\mathrm{B}}/\zeta_{\mathrm{hp}}$ exceeds the range of (4.30) there is localized mutual repulsion and

$$[\lambda]_{\text{equilibrium}} = \ln\left(\frac{\beta^2\zeta_{\mathrm{B}}}{(3+3\beta+\beta^2)\zeta_{\mathrm{hp}}}\right). \tag{4.32}$$

The variation of $[\lambda]_{\text{equilibrium}}$ with $\zeta_{\mathrm{B}}/\zeta_{\mathrm{hp}}$ for various values of β is shown in figure 9.

(ii) *Specified surface slope at boundary*

A similar analysis to that used in deriving equation (4.29) yields

$$\frac{H}{2S} = \frac{\beta(1+\beta+\frac{1}{3}\beta^2)\mathrm{e}^{-\lambda}E_1'E_2'}{\{(1+\beta)(1+\beta+\frac{1}{3}\beta^2)+\frac{1}{3}\beta^3\mathrm{e}^{-2\lambda}\}^2}, \tag{4.33}$$

where

$$E_1' = \phi_{\mathrm{B}}(1+\beta)+\zeta_{\mathrm{hp}}\beta\mathrm{e}^{-\lambda},$$
$$E_2' = \zeta_{\mathrm{hp}}(1+\beta+\tfrac{1}{3}\beta^2)-\tfrac{1}{3}\phi_{\mathrm{B}}\beta^2\mathrm{e}^{-\lambda}.$$

It follows that when

$$\frac{\phi_{\mathrm{B}}}{\zeta_{\mathrm{hp}}} < -\left(\frac{\beta}{1+\beta}\right), \tag{4.34}$$

there is mutual repulsion for all values of λ. When

$$-\left(\frac{\beta}{1+\beta}\right) < \frac{\phi_{\mathrm{B}}}{\zeta_{\mathrm{hp}}} < 0, \tag{4.35}$$

there is mutual attraction over the range $0 < \lambda < \lambda^*$, where

$$\lambda^* = \ln\left(\frac{-\beta\zeta_{\mathrm{hp}}}{(1+\beta)\phi_{\mathrm{B}}}\right), \tag{4.36}$$

and mutual repulsion where $\lambda > \lambda^*$. It follows that if $\lambda = \lambda^*$, the strip is in an *unstable* equilibrium state; any deviation from this state results in the strip moving to a stable equilibrium state where $[\lambda]_{\mathrm{equilibrium}} = 0$ or $[\lambda]_{\mathrm{equilibrium}} \to \infty$.

When

$$0 < \frac{\phi_{\mathrm{B}}}{\zeta_{\mathrm{hp}}} < \left(\frac{3+3\beta+\beta^2}{\beta^2}\right), \tag{4.37}$$

there is mutual attraction for all values of λ, so that $[\lambda]_{\mathrm{equilibrium}} = 0$, while if $\phi_{\mathrm{B}}/\zeta_{\mathrm{hp}}$ exceeds this range, there is localized repulsion and

$$[\lambda]_{\mathrm{equilibrium}} = \ln\left(\frac{\beta^2\phi_{\mathrm{B}}}{(3+3\beta+\beta^2)\zeta_{\mathrm{hp}}}\right). \tag{4.38}$$

The variation of $[\lambda]_{\mathrm{equilibrium}}$ with $\phi_{\mathrm{B}}/\zeta_{\mathrm{hp}}$ for various values of β is shown in figure 10.

(*e*) *Extensive mutual repulsion between floating objects*

The phenomenon of extensive mutual repulsion occurs only rarely between objects supported by surface tension. In its extreme form in which there is, in theory, mutual repulsion over the complete range $0 < \lambda < \infty$, it is restricted to certain pairs of infinitely long strips. For strips of finite length l_{s}, say, the range of mutual repulsion, namely $0 < \lambda < [\lambda]_{\mathrm{equilibrium}}$, is also finite but, unlike the localized mutual repulsion considered in § 4 *c*, $[\lambda]_{\mathrm{equilibrium}}$ is of the order of l_{s} rather than L.

The phenomenon is most simply displayed by consideration of the equilibrium of two identical, infinitely long strips with an eccentric centre of gravity ϵ, similar to those considered in § 4 *b* except that the heavier side of one strip now faces the lighter side of the other. For sufficiently small values of ϵ there is mutual attraction for all values of the gap λ, and in what follows we determine critical values of ϵ that mark, first, the onset of localized mutual repulsion and, second, the onset of the

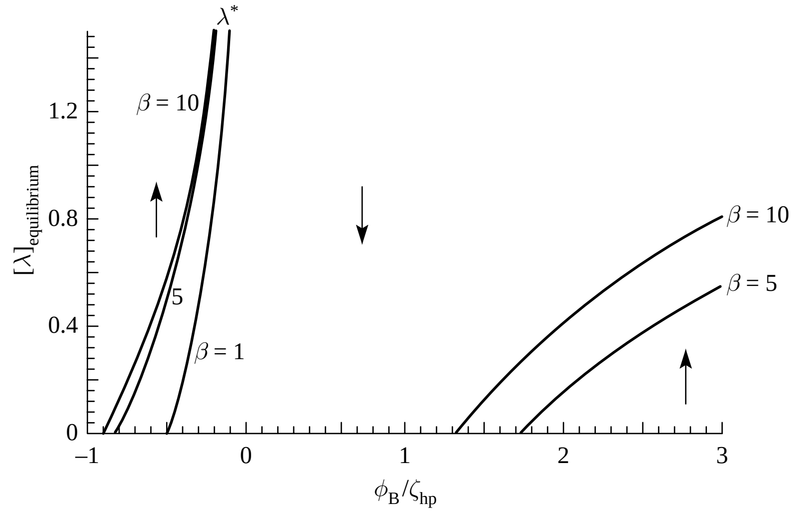

Figure 10. Variation of equilibrium gap with slope of liquid at boundary. Arrows indicate direction of force applied to strip away from equilibrium state. β = semi-width of strip, λ = distance from boundary, λ^* refers to unstable equilibrium state.

extreme form of extensive mutual repulsion. We adopt the notation of § 4 *c* so that the deflexions and slopes may again be expressed by equations (4.19), (4.20). The constants C_n are now determined from the equilibrium equations for each strip which are given by equation (4.4) with the suffices 1,2 replaced by a, b and c, d. Hence we find from equation (2.18) or (3.22),

$$H = \frac{W^2}{2S}\mathrm{e}^{-\lambda}\left\{\left(\frac{J}{J(1+\beta)+\beta G\mathrm{e}^{-\lambda}}\right)^2 - \left(\frac{3\epsilon\beta(1+\beta)}{J(1+\beta)-\beta G\mathrm{e}^{-\lambda}}\right)^2\right\}, \tag{4.39}$$

where G, J are as defined after equation (4.16). It follows that the force of mutual attraction is progressively reduced as $|\epsilon|$ increases from zero. Indeed, when

$$|\epsilon| = \frac{J\{J(1+\beta)-\beta G\}}{3\beta(1+\beta)\{J(1+\beta)+\beta G\}}, \tag{4.40}$$

the force H is positive for non-zero values of λ but zero at $\lambda = 0$. Thus, although the strips have coalesced there is then no force of mutual attraction between them. For increased values of $|\epsilon|$ there is localized mutual repulsion similar to that in § 4 *c*, and the equilibrium state is given by

$$[\lambda]_{\text{equilibrium}} = \ln\left(\frac{\beta G\{J+3\epsilon\beta(1+\beta)\}}{J(1+\beta)\{J-3\epsilon\beta(1+\beta)\}}\right). \tag{4.41}$$

Likewise, if

$$|\epsilon| \geqslant \epsilon^*, \tag{4.42}$$

where ϵ^* is defined in equation (4.11), there is extensive mutual repulsion because H is negative for all values of λ.

Finally, we note that although the variation of $[\lambda]_{\text{equilibrium}}$ with ϵ is continuous from the onset of localized mutual repulsion to extensive mutual repulsion, the form of equation (4.41) results in a nearly abrupt change in $[\lambda]_{\text{equilibrium}}$ as $|\epsilon| \to \epsilon^*$, as indicated in table 1 for the case in which $\beta = 5$ and $\tau \to 0$ so that $\epsilon^* = 0.4\dot{7}$.

Table 1. *Variation of equilibrium distance* $[\lambda]_{\text{equilibrium}}$ *with eccentricity of centre of gravity* ϵ *for shallow strips in which* $\beta = 5$

$\lvert\epsilon\rvert$	$0 \to 0.166$	0.2	0.3	0.4	0.47	0.477	$0.4\dot{7} \to 1$
$[\lambda]_{\text{equilibrium}}$	0	0.167	0.751	1.70	4.08	6.39	∞

(*f*) *Characteristic deflexion patterns in rafts*

The characteristic deflexion patterns in rafts of strips held together by the effects of surface tension are most simply displayed for rafts comprising identical, homogeneous strips of shallow rectangular cross-section so that we may take τ and ϵ zero.

Consider therefore the equilibrium of two adjacent strips whose edges are at junctions labelled $(n-1)$, n and $(n+1)$. We denote by ϕ^+_{n-1} and ϕ^-_n the values of ϕ at the edges of the strip bounded by $(n-1)$, n. The corresponding values of ϕ for the adjacent strip are accordingly ϕ^+_n and ϕ^-_{n+1} and, because of continuity of surface tension between the two strips and the reversal of the outward normals ν, we have

$$\phi^-_n = -\phi^+_n. \tag{4.43}$$

Now the equilibrium equations (4.4), for the two strips are

$$\left.\begin{aligned} \phi^+_{n-1} + \phi^-_n + \beta(\zeta_{n-1} + \zeta_n) &= W/S, \\ \beta(\phi^+_{n-1} - \phi^-_n) + (1 + \tfrac{1}{3}\beta^2)(\zeta_{n-1} - \zeta_n) &= 0, \end{aligned}\right\} \tag{4.44}$$

and

$$\left.\begin{aligned} \phi^+_n + \phi^-_{n+1} + \beta(\zeta_n + \zeta_{n+1}) &= W/S, \\ \beta(\phi^+_n - \phi^-_{n+1}) + (1 + \tfrac{1}{3}\beta^2)(\zeta_n - \zeta_{n+1}) &= 0. \end{aligned}\right\} \tag{4.45}$$

Equations (4.43)–(4.45) enable us to eliminate the ϕ terms and derive the following recurrence relation for the deflexions at successive junctions:

$$(\tfrac{2}{3} - \beta^{-2})\zeta_{n-1} + 2(\tfrac{4}{3} + \beta^{-2})\zeta_n + (\tfrac{2}{3} - \beta^{-2})\zeta_{n+1} = 4\zeta_{\text{hp}}, \tag{4.46}$$

where ζ_{hp} is defined by equation (4.28).

It may now be shown that the general solution of equation (4.46) is

$$\zeta_n = \zeta_{\text{hp}} + C_1\kappa^n + C_2\kappa^{-n}, \tag{4.47}$$

where

$$\kappa = \frac{3 + 4\beta^2 + 2\beta\{3(3+\beta^2)\}^{1/2}}{3 - 2\beta^2}, \tag{4.48}$$

and C_1, C_2 are constants to be determined from the boundary conditions.

(i) *Boundary conditions*

The deflexions ζ_n of the raft are symmetrical about its centre-line and when this condition is satisfied it is sufficient to consider the boundary condition at one edge, say the strip bounded by the junction $(N-1)$ and the free edge N. For this strip the equilibrium equations (4.4) are

$$\left.\begin{aligned} \phi^+_{N-1} + \phi_N + \beta(\zeta_{N-1} + \zeta_N) &= W/S, \\ \beta(\phi^+_{N-1} - \phi_N) + (1 + \tfrac{1}{3}\beta^2)(\zeta_{N-1} - \zeta_N) &= 0, \end{aligned}\right\} \tag{4.49}$$

from which ϕ^+_{N-1} may be eliminated. Now beyond the edge strip the deflexion is of the form

$$\zeta = \zeta_N \mathrm{e}^{-\nu}, \tag{4.50}$$

so that

$$\phi_N = \zeta_N, \tag{4.51}$$

and it follows that the boundary condition can be cast in the form

$$\zeta_N(3 + 6\beta + 4\beta^2) - \zeta_{N-1}(3 - 2\beta^2) = 6\beta^2\zeta_{\mathrm{hp}}. \tag{4.52}$$

(ii) *Forces of mutual attraction*

Equations (4.44) enable us to express ϕ^+_{n-1} and ϕ^-_n in terms of the deflexions ζ_{n-1} and ζ_n, and hence the horizontal force $H_{n-1,n}$ required for the equilibrium of the strip bounded by junctions $(n-1)$ and n may be expressed solely in terms of ζ_{n-1} and ζ_n. Thus we find

$$H_{n-1,n} = S(\zeta_{n-1} - \zeta_n)\{(1 + \tfrac{1}{3}\beta^2)\zeta_{\mathrm{hp}} - \tfrac{1}{6}\beta^2(\zeta_{n-1} + \zeta_n)\}. \tag{4.53}$$

Now the force $H_{N-1,N}$ required for equilibrium of the end strip is necessarily applied by the adjacent strip whose equilibrium requires that this force *plus* $H_{N-2,N-1}$ must be applied by the next strip, and so on. It follows that the horizontal compressive force F_n, say, between strips at a typical junction n is given by

$$\begin{aligned} F_n &= \sum_{n+1}^{N} H_{k-1,k} \\ &= S(\zeta_n - \zeta_N)\{(1 + \tfrac{1}{3}\beta^2)\zeta_{\mathrm{hp}} - \tfrac{1}{6}\beta^2(\zeta_n + \zeta_N)\}. \end{aligned} \tag{4.54}$$

In particular, the force F_0 at a central junction is given by

$$F_0 = S(\zeta_0 - \zeta_N)\{(1 + \tfrac{1}{3}\beta^2)\zeta_{\mathrm{hp}} - \tfrac{1}{6}\beta^2(\zeta_0 + \zeta_N)\}. \tag{4.55}$$

(iii) *Raft with even number of strips*

Here we consider a raft of $2N$ strips so that there are $(2N-1)$ junctions. If the central junction is labelled 0, then from symmetry the junction deflexions ζ_n satisfy the condition

$$\zeta_n = \zeta_{-n}, \tag{4.56}$$

so that we can write

$$\zeta_n = \zeta_{\mathrm{hp}} + A(\kappa^n + \kappa^{-n}), \tag{4.57}$$

where the constant A is determined from the boundary condition (4.52). Hence we find

$$\zeta_n/\zeta_{\mathrm{hp}} = 1 - \frac{\kappa^n + \kappa^{-n}}{\{1 + (1 + \frac{1}{3}\beta^2)^{1/2}\}\kappa^N + \{1 - (1 + \frac{1}{3}\beta^2)^{1/2}\}\kappa^{-N}}, \tag{4.58}$$

and the problem is formally solved.

(iv) *Raft with odd number of strips*

Consider now the case of $(2N-1)$ strips so that there are $(2N-2)$ junctions. We label these from $-(N-2)$ to $(N-1)$ so that the central strip is bounded by junctions 0 and 1. The deflexions ζ_n therefore satisfy the symmetry condition

$$\zeta_{-n} = \zeta_{n+1}, \tag{4.59}$$

and accordingly we can write

$$\zeta_n = \zeta_{\rm hp} + B(\kappa^n + \kappa^{-n+1}), \tag{4.60}$$

where the constant B is determined from the boundary condition (4.52). Hence we find

$$\zeta_n/\zeta_{\rm hp} = 1 - \frac{\kappa^n + \kappa^{-n+1}}{\{1+(1+\frac{1}{3}\beta^2)^{1/2}\}\kappa^N + \{1-(1+\frac{1}{3}\beta^2)^{1/2}\}\kappa^{-N+1}}. \tag{4.61}$$

As is to be expected, equation (4.61) shows that the central strip in the raft is not tilted, and comparison with equation (4.58) shows that the angles of tilt of the other strips and their edge deflexions are intermediate between those of rafts with 'adjacent' even numbers of strips.

(v) *Discussion*

In discussing the above results for both even and odd numbers of strips it is convenient to consider first the case in which $\beta \to (3/2)^{1/2}$, for then $\kappa \to \pm\infty$ and we have the simple result

$$\zeta_N/\zeta_{\rm hp} = 3 - \sqrt{6} \approx 0.5505, \tag{4.62}$$

and

$$\zeta_{N-1} = \zeta_{N-2} = \cdots = \zeta_0 = \zeta_{\rm hp}. \tag{4.63}$$

Thus only the outermost strips are tilted and they alone provide the horizontal force that holds the raft together whence, from equation (4.53):

$$H/S = \tfrac{1}{2}\zeta_{\rm hp}^2, \tag{4.64}$$

or, in dimensional terms,

$$H = W^2/12S. \tag{4.65}$$

When $\beta < (3/2)^{1/2}$, κ is positive and equations (4.58), (4.61) show that successive values of ζ_n increase monotonically, but by decreasing amounts, towards the centre. In contrast, when $\beta > (3/2)^{1/2}$, κ is negative and successive values of ζ_n oscillate about $\zeta_{\rm hp}$ by decreasing amounts down to the central deflexion.

As for the horizontal force $H_{n-1,n}$, it may readily be shown that the term in braces in equation (4.53) is necessarily positive and hence the direction of the force is determined by the sign of $(\zeta_{n-1} - \zeta_n)$. If $\beta < (3/2)^{1/2}$ it follows that $H_{n-1,n}$ is positive over the range $1 \leqslant n \leqslant N$ and hence the compressive force F_n between strips increases monotonically towards the centre. In contrast, if $\beta > (3/2)^{1/2}$ successive values of $H_{n-1,n}$ alternate in sign and the resultant forces F_n oscillate by decreasing amounts about F_0 as n varies from $(N-1)$ to 0. Examples of raft cross-sections with 4 strips with $\beta = 0.4(3/2)^{1/2}$, $(3/2)^{1/2}$ and $1.5(3/2)^{1/2}$ are shown in figure 11.

(*g*) *Influence of edge undulations on force of mutual attraction*

Consider an infinitely long, rigid strip of width 2β and weight per unit length W, whose wetted lower surface is given by

$$\left.\begin{aligned} \zeta' &= \Delta_c(\xi'/\beta)^2 \sin\omega_c\eta', && -\beta \leqslant \xi' \leqslant 0,\\ &= \Delta_d(\xi'/\beta)^2 \sin(\omega_d\eta' + k_d), && 0 \leqslant \xi' \leqslant \beta, \end{aligned}\right\} \tag{4.66}$$

where the primed coordinates are embedded in the strip, with ξ', η' in the plane tangential to the wetted surface along the mid-line where $\xi' = 0$. Such a strip has

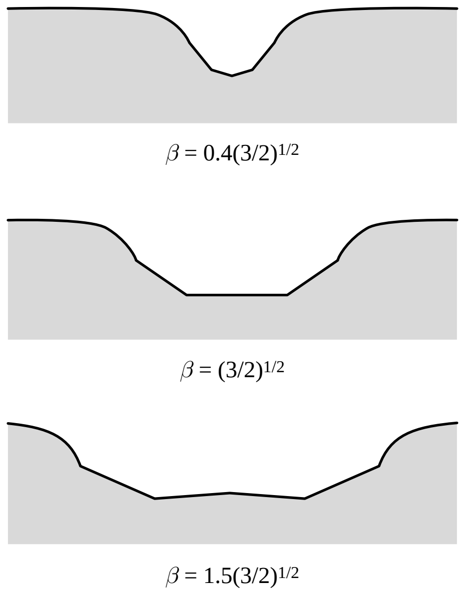

Figure 11. Cross-section of infinitely long rafts comprising four strips of different widths. Corresponding experimental results are given in figure 14.

wavy, pie-crust edges whose undulations may differ in magnitude, wavelength and phase. In discussing its equilibrium equations we note first that the volume $\mathcal{V}$, taken over an infinite length, is independent of the magnitude of the undulations nor, indeed, does this feature depend on the specified variation of ζ' with ξ', which was simply chosen for ease of visualization.

Consider now the edge at $\xi' = -\beta$, identified by suffix c, where the surface deflexion is given by

$$\zeta_c^*(\eta, \lambda) = \zeta_c^*(\lambda) + \varDelta_c \sin \omega_c \eta, \tag{4.67}$$

where $\zeta_c^*(\lambda)$ is a rigid-body displacement and λ is the distance to the facing edge of another strip where the surface deflexion is given by

$$\zeta_b^*(\eta, \lambda) = \zeta_b^*(\lambda) + \varDelta_b \sin(\omega_b \eta + k_b), \quad \text{say.} \tag{4.68}$$

It follows from equation (2.15) that the surface deflexion at a distance ν_c from edge$_c$ is given by

$$\begin{aligned}\zeta(\nu_c) = {} & \zeta_c^*(\lambda)(\cosh \nu_c - \coth \lambda \sinh \nu_c) + \zeta_b^*(\lambda)\{\cosh(\lambda - \nu_c) - \coth \lambda \sinh(\lambda - \nu_c)\} \\ & + \varDelta_c \sin \omega_c \eta (\cosh \varOmega_c \nu_c - \coth \varOmega_c \lambda \sinh \varOmega_c \nu_c) \\ & + \varDelta_b \sin(\omega_b \eta + k_b)\{\cosh \varOmega_b(\lambda - \nu_c) - \coth \varOmega_b \lambda \sinh \varOmega_b(\lambda - \nu_c)\},\end{aligned} \tag{4.69}$$

where

$$\varOmega_i = (1 + \omega_i^2)^{1/2}, \qquad i = b, c. \tag{4.70}$$

Now, along the edge at $\xi' = -\beta$,

$$\begin{aligned}\phi_c(\eta, \lambda) &= -\left(\frac{\partial \zeta}{\partial \nu_c}\right)_{\nu_c = 0}, \\ &= \zeta_c^*(\lambda) \coth \lambda - \zeta_b^*(\lambda) \operatorname{cosech} \lambda + \varOmega_c \varDelta_c \sin \omega_c \eta \coth \varOmega_c \lambda \\ &\quad - \varOmega_b \varDelta_b \sin(\omega_b \eta + k_b) \operatorname{cosech} \varOmega_b \lambda,\end{aligned} \tag{4.71}$$

and there are similar expressions for the edge at $\xi' = +\beta$, if it faces another strip. Now the equilibrium equations (3.19), (3.21) involve infinite integrals of $\phi(\eta, \lambda)$ and $\zeta^*(\eta, \lambda)$ whose sinusoidal components necessarily vanish. It follows that in an equilibrium state the rigid-body deflexions $\zeta_c^*(\lambda)$, $\zeta_d^*(\lambda)$ are independent of the magnitude of the undulations either of the strip itself or any adjacent strip; they are therefore the same as for a flat strip. However, the forces of mutual attraction or repulsion *are* influenced by edge waviness, and to investigate this in detail we first consider the form taken by equations (3.22), (3.23) for the strip defined by equation (4.66).

(i) *The forces of mutual attraction or repulsion*

The values of ϕ^2, etc., vary with η, and the forces of mutual attraction per unit length are most readily given in terms of average values. Thus equations (3.22), (3.23) become

$$\langle H_\xi/S\rangle = \langle Q_d\rangle - \langle Q_c\rangle, \tag{4.72}$$

where

$$Q_i = \tfrac{1}{2}\left[\Big\{\phi_i(\eta,\lambda)\Big\}^2 - \Big\{\zeta_i^*(\eta,\lambda)\Big\}^2 - \left\{\frac{\partial\zeta_i^*(\eta,\lambda)}{\partial\eta}\right\}^2\right], \qquad i = c, d$$

and

$$\langle H_\eta/S\rangle = \langle Q_d'\rangle - \langle Q_c'\rangle, \tag{4.73}$$

where

$$Q_i' = -\phi_i(\eta,\lambda)\frac{\partial\zeta_i^*(\eta,\lambda)}{\partial\eta}, \qquad i = c, d.$$

Let us suppose now that the edge at $\xi' = +\beta$, faces an infinite expanse of liquid. The surface deflexion at a distance ν_d from this edge is given by

$$\zeta(\nu_d) = \zeta_d^*(\lambda)e^{-\nu_d} + \Delta_d \sin(\omega_d\eta + k_d)e^{-\Omega_d\nu_d}, \tag{4.74}$$

and substitution into equations (4.72), (4.73) yields

$$\langle Q_d\rangle = \langle Q_d'\rangle = 0. \tag{4.75}$$

Note that these results could also have been deduced immediately from the arguments of § 3 *d* if the contour paths of integration extend from the outer edges to infinity. It follows that the force of mutual attraction between two such strips, depends only on the boundary conditions at their facing edges. If these are given by equations (4.67), (4.68), the surface deflexion between them is given by equation (4.69), whence we find

$$\begin{aligned}\langle H_\xi/S\rangle = {}& \tfrac{1}{2}\zeta_c^*(\lambda)\zeta_b^*(\lambda)\operatorname{sech}^2\tfrac{1}{2}\lambda - \tfrac{1}{2}\{\zeta_c^*(\lambda) - \zeta_b^*(\lambda)\}^2\operatorname{cosech}^2\lambda \\ & -\tfrac{1}{4}(\Omega_c\Delta_c\operatorname{cosech}\Omega_c\lambda)^2 - \tfrac{1}{4}(\Omega_b\Delta_b\operatorname{cosech}\Omega_b\lambda)^2 \\ & +\Omega_c\Omega_b\Delta_c\Delta_b\coth\Omega_c\lambda\operatorname{cosech}\Omega_b\lambda\langle\sin\omega_c\eta\sin(\omega_b\eta + k_b)\rangle,\end{aligned} \tag{4.76}$$

where $\zeta_c^*(\lambda), \zeta_b^*(\lambda)$ are given by the methods of § 4 *b*–*e* for flat strips. Note that unless $\omega_b = \omega_c$, the last term in equation (4.76) is zero, whatever the value of k_b. In what follows we assume that $\omega_b = \omega_c$, noting that

$$\langle\sin\omega_c\eta\sin(\omega_c\eta + k_b)\rangle = \tfrac{1}{2}\cos k_b. \tag{4.77}$$

Likewise, it may be shown that $\langle H_\eta\rangle$ is zero unless $\omega_b = \omega_c$, when

$$\langle H_\eta/S\rangle = -\tfrac{1}{2}\omega_c\Omega_c\Delta_c\Delta_b\sin k_b\operatorname{cosech}\Omega_c\lambda. \tag{4.78}$$

(ii) *Strips with mirror-image symmetry*

When $k_b = 0$, $\Delta_b = \Delta_c$, $\omega_b = \omega_c$ and $\zeta_b^*(\lambda) = \zeta_c^*(\lambda)$, we find

$$\langle H_\xi / S \rangle = \tfrac{1}{2}\{\zeta_c^*(\lambda)\,\mathrm{sech}\,\tfrac{1}{2}\lambda\}^2 + \tfrac{1}{4}(\Omega_c \Delta_c\,\mathrm{sech}\,\tfrac{1}{2}\Omega_c\lambda)^2, \tag{4.79}$$

so that the force of mutual attraction is increased by the presence of undulations. Note that in terms of the surface deflexion ζ_{cl} along the midway line between the strips, equation (4.79) can be expressed as

$$\langle H_\xi / S \rangle = \tfrac{1}{2}\langle \zeta_{\mathrm{cl}}^2 + (\mathrm{d}\zeta_{\mathrm{cl}}/\mathrm{d}\eta)^2 \rangle, \tag{4.80}$$

in accord with the arguments of § 3 *e*. The maximum value of $\langle H_\xi \rangle$ occurs when the strips have coalesced;

$$\langle H_\xi / S \rangle_{\max} = \tfrac{1}{2}\{\zeta_c^*(0)\}^2 + \tfrac{1}{4}\Omega_c^2 \Delta_c^2. \tag{4.81}$$

There are, however, limits to the values that Ω_c and Δ_c may take because of restrictions on the magnitude of the surface slopes and, for example, a possible requirement that the wetted lower surface is below the remote surface level for all values of λ. Nevertheless, for values of $\zeta_c^*(0) < 0.2$, say, edge undulations can more than double the force of mutual attraction.

(iii) *The phenomenon of mutual alignment*

Consider again the case just considered, except that the mirror-image symmetry is broken by the presence of a phase difference k_b. The force of mutual attraction can now be expressed in the form:

$$\begin{aligned}\langle H_\xi / S \rangle = \tfrac{1}{2}\{\zeta_c^*(\lambda)\,\mathrm{sech}\,\tfrac{1}{2}\lambda\}^2 + \tfrac{1}{4}\Omega_c^2 \Delta_c^2\{&\mathrm{sech}^2\,\tfrac{1}{2}\Omega_c\lambda \\ &-2\,\mathrm{cosech}^2\,\Omega_c\lambda \cosh \Omega_c\lambda(1-\cos k_b)\},\end{aligned} \tag{4.82}$$

where it is seen that unless $\cos k_b = 1$, the above expression becomes negative and unbounded as $\lambda \to 0$, indicating a high degree of repulsion. This singular behaviour stems from a localized breakdown in linear theory which requires the surface slope to become vertical as $\lambda \to 0$. In practice it means that *if the phase difference k_b does not change*, a state of equilibrium exists where λ is such that $\langle H_\xi / S \rangle$ is zero, but this equilibrium state requires the application of a force $\langle H_\eta \rangle$, given by equation (4.78). If this equilibrating force is removed, it results in a relative shearing displacement of the strips, leading to a reduction of the phase angle k_b. It follows that the only stable state for freely supported strips is that considered previously with $k_b = 0$. This phenomenon of mutual alignment is, of course, also applicable to finite objects, and in the context of evolution this could have some biological advantage.

5. Upright circular cylinder with arbitrary centre of gravity

As a further example in the application of the equilibrium equations of § 3, we consider the equilibrium of an upright, circular cylinder with arbitrary centre of gravity, the boundary of whose flat lower surface meets the surface of the liquid. The cylinder has a radius ρ_0 and weight $\mathcal{W}$, whose centre of gravity is at

$$(\xi'_{\mathrm{cg},\mathcal{W}},\ \eta'_{\mathrm{cg},\mathcal{W}},\ \zeta'_{\mathrm{cg},\mathcal{W}}) = (\epsilon\rho_0,\ 0,\ -\tau). \tag{5.1}$$

Note that although we focus attention on this particular shape, the analysis applies to any object that has the same weight, the same centre of gravity, the same flat lower surface and the same contact with the liquid.

We search for a solution that satisfies equation (2.15) in the form

$$\zeta = AK_0(\rho) + BK_1(\rho)\cos\theta, \tag{5.2}$$

where K_0 and K_1 are modified Bessel functions and A, B are constants. Hence

$$\phi = -\left(\frac{\partial\zeta}{\partial\rho}\right)_{\rho_0} = AK_1(\rho_0) + B\{K_0(\rho_0) + K_1(\rho_0)/\rho_0\}\cos\theta, \tag{5.3}$$

and the deflexion of the flat lower surface of the cylinder therefore lies in the tilted plane

$$\zeta = \zeta_0 + \alpha_\xi \xi, \tag{5.4}$$

where

$$\zeta_0 = AK_0(\rho_0), \qquad \alpha_\xi = BK_1(\rho_0)/\rho_0. \tag{5.5}$$

Vertical equilibrium, see equation (3.19), now yields

$$A = \frac{\mathcal{W}}{\pi\rho_0\{\rho_0 K_0(\rho_0) + 2K_1(\rho_0)\}}. \tag{5.6}$$

In discussing moment equilibrium it is convenient to introduce γ_{C}, the ratio of the average density of the cylinder to that of the liquid, so that if τ is at the mid-height of the cylinder,

$$\gamma_{\mathrm{C}} = \frac{\mathcal{W}}{2\pi\rho_0^2\tau}. \tag{5.7}$$

Substitution into equation (3.21) now yields

$$B = \frac{\epsilon\mathcal{W}}{\pi[\rho_0 K_0(\rho_0) + \{2(1-\tau^2\gamma_{\mathrm{C}}) + \frac{1}{4}\rho_0^2\}K_1(\rho_0)]}. \tag{5.8}$$

The constant B is proportional to the angle of tilt of the cylinder and it follows from equation (5.8) that if

$$\tau^2\gamma_{\mathrm{C}} \ll 1 + \frac{\rho_0^2}{8} + \frac{\rho_0 K_0(\rho_0)}{2K_1(\rho_0)}, \tag{5.9}$$

we can ignore the destabilizing effect caused by the centre of gravity being above the plane of contact with the liquid, and the cylinder can be regarded as a shallow disc.

(a) *Tilting of the cylinder base above the remote liquid level*

Here we determine the critical value of ϵ, ϵ^* say, above which parts of the lighter side of the cylinder base tilt above the remote level of the liquid. If we take ϵ^* positive, it is therefore determined by the vanishing of the edge deflexion at $\theta = -\pi$, whence

$$AK_0(\rho_0) = BK_1(\rho_0), \tag{5.10}$$

so that from equations (5.6) and (5.8),

$$\epsilon^* = \frac{K_0(\rho_0)}{\rho_0 K_1(\rho_0)}\left(1 + \frac{(\rho_0^2 - 8\tau^2\gamma_{\mathrm{C}})K_1(\rho_0)}{4\{\rho_0 K_0(\rho_0) + 2K_1(\rho_0)\}}\right). \tag{5.11}$$

Note that although ϵ^* is necessarily less than unity for circular cylinders there is always a value of $\tau^2\gamma_{\mathrm{C}}$ which satisfies equation (5.11), whatever the value of ρ_0. In contrast, for shallow circular discs that satisfy the inequality (5.9), such situations can only occur if $\rho_0 > 0.634$.

(*b*) *Toppling of a homogeneous, upright, circular cylinder*

From equations (5.5), (5.8) we see that, in general, if ϵ is zero so too is the angle of tilt α_ξ. But this is not always the case, because if the denominator in the expression for B is zero, the expression for α_ξ becomes indeterminate. This is indicative of a toppling instability which therefore occurs if

$$\tau^2\gamma_{\mathrm{C}} = 1 + \frac{\rho_0^2}{8} + \frac{\rho_0 K_0(\rho_0)}{2K_1(\rho_0)}. \tag{5.12}$$

6. Arbitrarily shaped objects with planar lower surfaces

(*a*) *The equilibrium equations*

Problems in which the surface of the liquid meets the boundary of the flat lower surface of an object, such as those discussed in § 4 *a*–*f* and § 5, are the simplest to analyse because the curve $\mathcal{C}$ is known, and $\mathcal{V}$ and other terms associated with the effects of hydrostatic pressure are readily determinable because $\mathcal{C}^*$ lies in a plane,

$$\zeta = \zeta_0 + \alpha_\xi \xi + \alpha_\eta \eta. \tag{6.1}$$

The equilibrium equations (3.19)–(3.21) are further simplified by aligning the axes so that the origin is at the centroid of the area $\mathcal{A}$ that is bounded by $\mathcal{C}$, and $O\xi$, $O\eta$ lie along the principal axes of $\mathcal{A}$, so that

$$\int \xi \,\mathrm{d}\mathcal{A} = \int \eta \,\mathrm{d}\mathcal{A} = \int \xi\eta \,\mathrm{d}\mathcal{A} = 0. \tag{6.2}$$

Referring now to the various terms in equations (3.19)–(3.21) we find,

$$\begin{aligned}\mathcal{V} &= \int \zeta \,\mathrm{d}\mathcal{A},\\ &= \zeta_0 \mathcal{A}.\end{aligned} \tag{6.3}$$

Likewise,

$$\oint \zeta^* \sin\Theta \,\mathrm{d}\sigma = \alpha_\eta \mathcal{A}, \qquad \oint \zeta^* \cos\Theta \,\mathrm{d}\sigma = \alpha_\xi \mathcal{A}. \tag{6.4}$$

To determine $\eta_{\mathrm{cg},\mathcal{V}}$, for example, we solve for

$$\int (\eta - \eta_{\mathrm{cg},\mathcal{V}})\zeta \,\mathrm{d}\mathcal{A} = 0, \tag{6.5}$$

whence

$$\eta_{\mathrm{cg},\mathcal{V}} = \frac{\alpha_\eta \mathcal{I}_\xi}{\zeta_0 \mathcal{A}}, \tag{6.6}$$

where $\mathcal{I}_\xi$ is the second moment of area about $O\xi$, and similarly

$$\xi_{\mathrm{cg},\mathcal{V}} = \frac{\alpha_\xi \mathcal{I}_\eta}{\zeta_0 \mathcal{A}}. \tag{6.7}$$

Finally, we express $\xi_{\mathrm{cg},\mathcal{W}}$, $\eta_{\mathrm{cg},\mathcal{W}}$ in terms of primed coordinates that are embedded in the object, where $O'\xi'$, $O'\eta'$ are in the plane of the wetted surface and aligned so that their vertical projections onto the horizontal plane coincide with $O\xi$, $O\eta$. Hence we find

$$\xi_{\mathrm{cg},\mathcal{W}} = \xi'_{\mathrm{cg},\mathcal{W}} - \alpha_\xi \zeta'_{\mathrm{cg},\mathcal{W}}, \qquad \eta_{\mathrm{cg},\mathcal{W}} = \eta'_{\mathrm{cg},\mathcal{W}} - \alpha_\eta \zeta'_{\mathrm{cg},\mathcal{W}}. \tag{6.8}$$

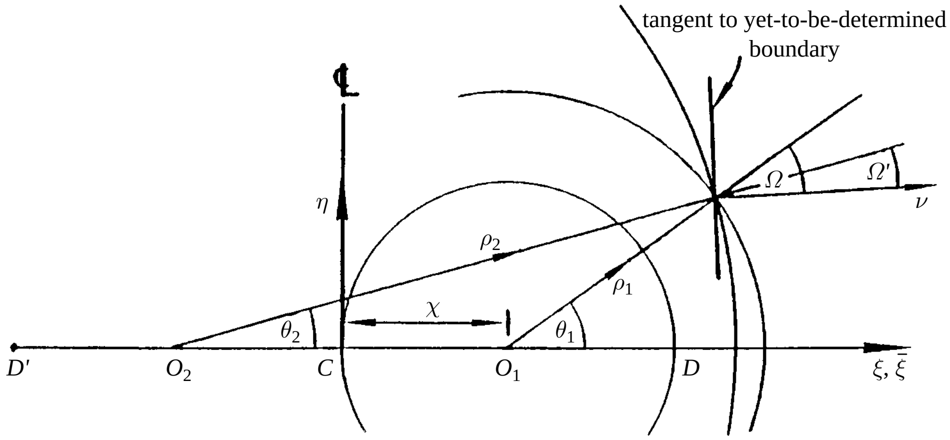

Figure 12. Notation for discs of oval shape.

Simplifications also arise in determining the forces of mutual attraction because the contribution of the hydrostatic pressure can be derived directly and more simply from the pressures applied over the area $\mathcal{A}$. Thus we find

$$\oint \tfrac{1}{2}\zeta^{*^2} \cos\Theta \,\mathrm{d}\sigma = \alpha_\xi \zeta_0 \mathcal{A}, \qquad \oint \tfrac{1}{2}\zeta^{*^2} \sin\Theta \,\mathrm{d}\sigma = \alpha_\eta \zeta_0 \mathcal{A}. \tag{6.9}$$

Likewise,

$$\oint \tfrac{1}{2}\rho^* \zeta^{*^2} \sin(\theta - \Theta)\,\mathrm{d}\sigma = \alpha_\xi \alpha_\eta (\mathcal{I}_\xi - \mathcal{I}_\eta). \tag{6.10}$$

Finally, in the contribution of surface tension, we have

$$\frac{\mathrm{d}\zeta^*}{\mathrm{d}\sigma} = \alpha_\eta \cos\Theta - \alpha_\xi \sin\Theta. \tag{6.11}$$

Simplified expressions for $\mathcal{H}$ between mirror-image objects are given in § 3 *e*. For circular discs similar arguments apply for other arrays including, in particular, the stable clover-leaf array of three touching discs. Problems of this nature may be considered in a later paper but in what follows we present an inverse type of analysis that yields a numerical solution for pairs of touching discs of oval shape, the boundaries of whose flat lower surfaces meet the surface of the liquid.

(*b*) *Inverse method of analysis for touching pairs of oval discs*

We assume that the pair of touching discs, whose shape is at present undetermined, exhibit symmetry both about their common tangent at the point of contact and about a horizontal line orthogonal to this tangent. With the notation of figure 12, C is the point of contact and D, D' are on each boundary diametrically opposite C; the points O_1, O_2 are at the centres of the diameters CD, CD', each of length 2χ. We adopt polar coordinates ρ_1, θ_1 and ρ_2, θ_2 centred on O_1 and O_2 respectively but we focus attention on the right-hand disc.

Guided by the results of § 5 and to satisfy all the symmetry conditions, we take the deflexion of the liquid surface to be given by

$$\zeta = \zeta_1 + \zeta_2, \tag{6.12}$$

where

$$\zeta_1 = A\{K_0(\rho_1) + \mu K_1(\rho_1)\cos\theta_1\},$$
$$\zeta_2 = A\{K_0(\rho_2) - \mu K_1(\rho_2)\cos\theta_2\},$$

and A, μ are constants. Equation (6.12) may be expressed solely in terms of ρ_1, θ_1 coordinates via the relations

$$\rho_2 = (4\chi^2 + 4\chi\rho_1\cos\theta_1 + \rho_1^2)^{1/2}, \qquad \theta_2 = \arctan\left(\frac{\rho_1\sin\theta_1}{2\chi + \rho_1\cos\theta_1}\right). \tag{6.13}$$

Now C and D are in the right-hand disc which therefore lies in the tilted plane

$$\zeta_\mathrm{d},\ \text{say}, = \tfrac{1}{2}(\zeta_C + \zeta_D) + \alpha_\xi\rho_1\cos\theta_1, \tag{6.14}$$

where

$$\zeta_C = 2A\{K_0(\chi) - \mu K_1(\chi)\},$$
$$\zeta_D = A[K_0(\chi) + K_0(3\chi) + \mu\{K_1(\chi) - K_1(3\chi)\}],$$

and the angle of tilt is given by

$$\alpha_\xi = \frac{\zeta_D - \zeta_C}{2\chi}. \tag{6.15}$$

It follows that the boundary of this disc for given values of χ and μ is at the intersection of this plane with the deflected liquid surface, i.e.

$$\zeta_\mathrm{d} = \zeta, \tag{6.16}$$

which may be determined numerically as θ_1 varies incrementally from 0 to π. Such discs can be made to satisfy the condition of vertical equilibrium by appropriate choice of the constant A. But the condition of moment equilibrium specifies a value for $(\xi'_{\mathrm{cg},\mathcal{W}} - \alpha_\xi\zeta'_{\mathrm{cg},\mathcal{W}})$. It follows from equation (6.8) that each such solution is appropriate to discs with a specified value of $\xi_{\mathrm{cg},\mathcal{W}}$ in the equilibrium state. The range of valid solutions is, however, limited because $\zeta'_{\mathrm{cg},\mathcal{W}}$ must be negative, while too large a value of $|\zeta'_{\mathrm{cg},\mathcal{W}}|$ results in a toppling instability as discussed in § 5 *b*. Here we demonstrate the method by treating the simple case of shallow discs of uniform density so that we may take $\zeta'_{\mathrm{cg},\mathcal{W}}$ and $\xi'_{\mathrm{cg},\mathcal{W}}$ to be zero. Details of the numerical analysis are given in Appendix C. For a given value of χ these first involve the determination by trial-and-error of the constant μ to satisfy the condition of zero $\xi_{\mathrm{cg},\mathcal{W}}$. The constant A is then given by equation (3.19) and, from equation (3.34), the force of mutual attraction is given by

$$\mathcal{H}_\xi = 4\chi A^2\int_0^{\frac{1}{2}\pi}[\{K_0(\rho)\sec\theta - \mu K_1(\rho)\}^2 + \sin^2\theta\{K_1(\rho)\sec\theta - \mu K_2(\rho)\}^2]\,\mathrm{d}\theta, \tag{6.17}$$

where

$$\rho = \chi\sec\theta.$$

Some numerical results are shown in table 2.

The corresponding shapes of the oval discs are shown in figure 13. Note that these equilibrium configurations are unstable because a small but opposing rotation (i.e. rolling) of the discs introduces equal and opposite torques to each disc, stemming initially from the now offset angle of the force $\mathcal{H}_\xi$. Such rotations increase until

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Table 2. *Deflexion constants μ, A and force of mutual attraction $\mathcal{H}_\xi$ between touching oval discs of weight $\mathcal{W}$*

major diameter of oval, 2χ	μ	A	$\mathcal{H}_\xi$
2	0.0564	$0.1869\mathcal{W}$	$0.0263\mathcal{W}^2$
4	0.1207	$0.2803\mathcal{W}$	$0.0045\mathcal{W}^2$

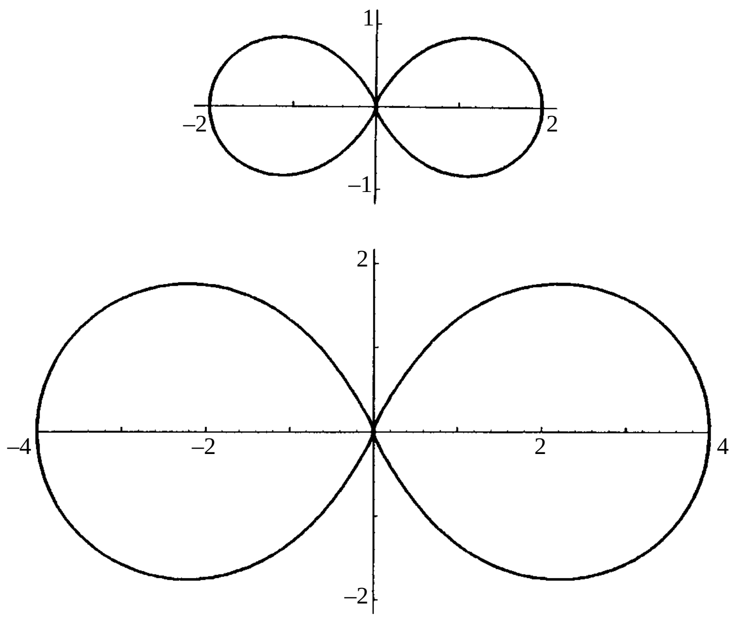

Figure 13. Touching uniform discs of oval shape (major diameter = 2, 4).

a further configuration is reached where the torques are again zero, which occurs approximately when the line of action of the force of mutual attraction, now $\mathcal{H}_{\xi,\mathrm{s}}$ say, next passes through the centres of gravity — rather like the behaviour of an egg on a horizontal surface. Detailed analysis of this is not attempted here but rough estimates suggest that for both cases $\mathcal{H}_{\xi,\mathrm{s}}$ will exceed $\mathcal{H}_\xi$ by about 20%. Further families of touching discs may, of course, be analysed by taking additional terms in the expressions for ζ_1 and ζ_2 including, for example, K_0 functions whose origins are symmetrically disposed about the common tangent at the point of contact.

7. Flexible discs

If there has been a coalescence of numerous objects whose linear dimensions are small in comparison with L, the resulting patch may be analysed as a continuum of infinitely many vanishingly small objects with the same overall weight per unit area. Such patches have negligible out-of-surface rigidity and will be referred to as flexible discs (or strips) to distinguish them from the rigid objects considered previously. The equations governing the surface deflexion are

$$\left.\begin{aligned} \zeta - \nabla^2\zeta &= \zeta_{\rm hp}, \qquad \text{under a flexible disc,} \\ \zeta - \nabla^2\zeta &= 0, \qquad \text{away from a disc,} \end{aligned}\right\} \tag{7.1}$$

where, as in §4,

$$\zeta_{\rm hp} = (\text{weight per unit area})/(\gamma g L).$$

If there is more than one flexible disc, the deflexion over the entire surface is the sum of deflexions due to each disc, in contrast to the case of rigid discs where neighbouring discs inhibit any curvature of the surface. It is this feature that is responsible for the relative simplicity of this class of problem.

(*a*) *Equilibrium of single, flexible strip*

The deflexion of a single, flexible strip of width 2β is readily found from equation (7.1) by ensuring continuity of surface deflexion and slope at the edges. Thus, we find

$$\left.\begin{aligned} \zeta &= \zeta_{\rm hp}(1 - \mathrm{e}^{-\beta}\cosh\xi), \quad -\beta \leqslant \xi \leqslant \beta, \\ \zeta &= \zeta^* \mathrm{e}^{-\nu}, \qquad\qquad\qquad |\xi| > \beta, \end{aligned}\right\} \tag{7.2}$$

where

$$\zeta^* = \tfrac{1}{2}\zeta_{\rm hp}(1 - \mathrm{e}^{-2\beta}), \tag{7.3}$$

and ν is the distance away from an edge. These results can also be derived by a limiting process from equation (4.58) as $N \to \infty$ for a raft comprising $2N$ rigid strips each of width δ, so that

$$\begin{aligned} \delta &= \beta/N, \\ &= \xi/n. \end{aligned} \tag{7.4}$$

Likewise, from equation (4.54), the compressive forces within a flexible strip are given by

$$F_\xi/S = \zeta_{\rm hp}^2 \mathrm{e}^{-\beta}(\cosh\beta - \cosh\xi), \tag{7.5}$$

and we note that across the centre-line of the strip,

$$F_0/S = \tfrac{1}{2}\zeta_{\rm hp}^2(1 - \mathrm{e}^{-\beta})^2. \tag{7.6}$$

Note, finally, that equation (7.5) may also be derived by regarding the flexible strip as supported on a frictionless but curved surface. The compressive forces F_ξ are then identified as those required to prevent elements of the strip from sliding towards its centre-line, like the buffer forces between stationary railway trucks on a line across a saucer-shaped depression. In the present scenario, an elemental strip of unit length and width $\delta\xi$ has a weight equal to $\zeta_{\rm hp}\delta\xi$ and, because of the slope of the surface, the element requires a horizontal force δF_ξ for equilibrium, where

$$\frac{\delta F_\xi}{S} = \zeta_{\rm hp}\frac{\mathrm{d}\zeta}{\mathrm{d}\xi}\delta\xi. \tag{7.7}$$

Now F_ξ is zero at $\xi = \beta$, and hence integration of equation (7.7) yields

$$F_\xi/S = \zeta_{\rm hp}(\zeta - \zeta^*), \tag{7.8}$$

in accord with equations (7.2), (7.5).

(b) Mutual attraction between two, unequal, flexible strips

Consider two, unequal, flexible strips, identified by suffices a, b, that are separated by a distance λ. The surface deflexion due solely to each strip is given by equations (7.2), (7.3), with appropriate suffix and appropriate origin for the ξ coordinate. The total deflexion of the surface is the sum of these components. Thus, focusing attention on strip b, say, the resultant values of the surface deflexion and slope at the edge facing strip a, identified by suffix 1, are given by

$$\zeta^*_{b,1} = \zeta^*_b + \mathrm{e}^{-\lambda}\zeta^*_a, \qquad \phi_{b,1} = \zeta^*_b - \mathrm{e}^{-\lambda}\zeta^*_a. \tag{7.9}$$

There is no need to determine the corresponding values at the outermost edge (suffix 2, say) because

$$\zeta^*_{b,2} = \phi_{b,2}, \tag{7.10}$$

and these terms cancel each other in the expression for H_{ab}. Thus, from equations (2.18), (7.9),

$$H_{ab}/S = \tfrac{1}{2}\zeta_{\mathrm{hp},a}\zeta_{\mathrm{hp},b}(1-\mathrm{e}^{-2\beta_a})(1-\mathrm{e}^{-2\beta_b})\mathrm{e}^{-\lambda}, \tag{7.11}$$

the symmetry of this expression reflecting the fact that $H_{ab} = H_{ba}$.

It follows that for flexible strips there is always mutual attraction, in contrast to the case of unequal rigid strips discussed in § 4 *c*. Furthermore, for large values of λ equations (4.22) and (7.11) give identical results as β_1, $\beta_2 \to 0$, but for large values of β_1, β_2,

$$H_{\text{rigid strips}} \to 4H_{\text{flexible strips}}. \tag{7.12}$$

Note, finally, that the force of mutual attraction within a single flexible strip, see equation (7.5), can also be derived from equation (7.11) by regarding ξ as the junction between two touching strips of unequal widths, i.e. by taking

$$\zeta_{\mathrm{hp},a} = \zeta_{\mathrm{hp},b}, \quad \beta_a = \tfrac{1}{2}(\beta+\xi), \quad \beta_b = \tfrac{1}{2}(\beta-\xi), \quad \lambda = 0. \tag{7.13}$$

(c) Equilibrium of single, flexible, circular disc

When there is a single, flexible, circular disc of radius ρ_a and uniform weight per unit area, we search for a solution of equations (7.1) in the form,

$$\left.\begin{aligned} \zeta &= \zeta_{\mathrm{hp},a}\{1 - A_a I_0(\rho)\}, & 0 \leqslant \rho \leqslant \rho_a, \\ &= \zeta_{\mathrm{hp},a} B_a K_0(\rho), & \rho > \rho_a, \end{aligned}\right\} \tag{7.14}$$

where the modified Bessel functions $I_0(\rho)$, $K_0(\rho)$ satisfy conditions as $\rho \to 0, \infty$, and the constants A_a, B_a are chosen to ensure continuity of deflexion and slope at ρ_a. Thus we find

$$A_a = K_1(\rho_a)/G_a, \qquad B_a = I_1(\rho_a)/G_a, \tag{7.15}$$

where

$$G_a = K_0(\rho_a)I_1(\rho_a) + K_1(\rho_a)I_0(\rho_a).$$

In particular, the deflexion ζ^*_a and radial slope ϕ_a at the boundary of the disc are given by

$$\zeta^*_a = \zeta_{\mathrm{hp},a}B_aK_0(\rho_a), \qquad \phi_a = \zeta_{\mathrm{hp},a}B_aK_1(\rho_a). \tag{7.16}$$

The compressive forces within a flexible circular disc are most readily determined by regarding the disc as supported on a frictionless but curved surface, as discussed in § 7 *a*. We assume that the continuum of vanishingly small objects that comprise a flexible circular disc has negligible resistance to shearing forces in its plane. It follows that at every point the principal compressive forces are of equal magnitude, so that

$$F_\rho = F_\theta, \tag{7.17}$$

where F_ρ and F_θ are the radial and hoop compressions per unit length. These forces are independent of θ but vary with ρ.

Consider now an element of the disc bounded by radii at θ and $\theta + \delta\theta$, and by circular arcs of radii ρ, $\rho+\delta\rho$. The area of this element is $\rho\delta\rho\delta\theta$ and hence its weight is $\zeta_{\mathrm{hp}}\rho\delta\rho\delta\theta$. Because of the radial slope of the surface, this element requires a resultant radial force to maintain its equilibrium. Thus, resolving in the radial direction yields,

$$\{\rho F_\rho - (\rho + \delta\rho)(F_\rho + \delta F_\rho) - F_\theta \delta\rho\}\delta\theta + \zeta_{\mathrm{hp}}\rho\frac{\partial\zeta}{\partial\rho}\delta\rho\delta\theta = 0, \tag{7.18}$$

whence, from equation (7.17), and dropping the suffices ρ and θ,

$$\frac{\partial F}{\partial\rho} = \zeta_{\mathrm{hp}}\frac{\partial\zeta}{\partial\rho}. \tag{7.19}$$

Now F is zero at the boundary, so that

$$\begin{aligned} F &= \zeta_{\mathrm{hp}}(\zeta - \zeta^*), \\ &= \zeta_{\mathrm{hp},a}^2 A_a\{I_0(\rho_a) - I_0(\rho)\}. \end{aligned} \tag{7.20}$$

(*d*) *Mutual attraction between two, unequal, flexible, circular discs*

If there are two, flexible, circular discs identified by suffices a, b, say, the resultant deflexion of the surface is the sum of the deflexions due to a and b. Thus, if the centres of the discs lie on the ξ-axis and we focus attention on disc a in determining the force of mutual attraction, we need to know the changes in ζ_a^*, ϕ_a and $\mathrm{d}\zeta_a^*/\mathrm{d}\sigma$ caused by disc b. In this connection we note that a point (ρ_a, θ_a) on the boundary of disc a corresponds to a point (ρ_{ba}, θ_{ba}), say, referred to an origin at the centre of disc b, where

$$\rho_{ba} = (\chi^2 + 2\chi\rho_a\cos\theta_a + \rho_a^2)^{1/2}, \quad \theta_{ba} = \arctan\left(\frac{\rho_a\sin\theta_a}{\chi + \rho_a\cos\theta_a}\right), \tag{7.21}$$

where

$$\chi = \rho_a + \rho_b + \lambda,$$

and λ is the gap between the two discs. It follows that at a point (ρ_a, θ_a) on the boundary of disc a, the deflexion caused by b is $\zeta_{\mathrm{hp},b}B_bK_0(\rho_{ba})$, and there is a surface slope $\zeta_{\mathrm{hp},b}B_bK_1(\rho_{ba})$ directed at the angle θ_{ba}, which can be resolved into components that are normal and tangential to disc a. Thus if we introduce the following terms,

$$\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \\ f_5 & f_6 \end{pmatrix} = \begin{pmatrix} \zeta_{\mathrm{hp},a}B_aK_0(\rho_a) & \zeta_{\mathrm{hp},b}B_bK_0(\rho_{ba}) \\ \zeta_{\mathrm{hp},a}B_aK_1(\rho_a) & \zeta_{\mathrm{hp},b}B_bK_1(\rho_{ba}) \\ f_4\sin(\theta_a - \theta_{ba}) & f_4\cos(\theta_a - \theta_{ba}) \end{pmatrix}, \tag{7.22}$$

the resultant deflexions and slopes around the boundary of disc a are given by

$$\zeta^* = f_1 + f_2, \qquad \phi = f_3 + f_6, \qquad \mathrm{d}\zeta^*/\mathrm{d}\sigma = f_5, \tag{7.23}$$

and, from equation (3.22), the force of mutual attraction is given by

$$\mathcal{H} = \rho_a\oint[\tfrac{1}{2}\{(f_3 + f_6)^2 - (f_1 + f_2)^2 - f_5^2\}\cos\theta_a + f_5(f_3 + f_6)\sin\theta_a]\,\mathrm{d}\theta_a. \tag{7.24}$$

Table 3. *Numerical values of* $\mathcal{H}$ *for unequal and equal discs*

ρ_a	ρ_b	$\zeta_{\mathrm{hp},a}$	$\zeta_{\mathrm{hp},b}$	λ	$\mathcal{H}$
0.5	1	1	2	0.5	0.128087
1	0.5	2	1	0.5	0.128087
1	2	1	2	1	0.282045
2	1	2	1	1	0.282045
1	1	1	1	0.5	0.148290
1	1	1	1	1	0.080589

Table 3 shows some results of the numerical integration of equation (7.24) for a set of values of ρ_a, ρ_b, $\zeta_{\mathrm{hp},a}$, $\zeta_{\mathrm{hp},b}$ and λ, corresponding to $\mathcal{H}_{ab}$ and with suffices a and b interchanged, corresponding to $\mathcal{H}_{ba}$. A numerical check on the validity of equation (7.24) is provided by the fact that $\mathcal{H}_{ab} = \mathcal{H}_{ba}$. Also shown in the last two rows are the values for $\mathcal{H}$ obtained by the alternative use of equation (3.34) for equal discs, which yields

$$\mathcal{H} = 4\zeta_{\mathrm{hp}}^2 B_a^2(\rho_a + \tfrac{1}{2}\lambda)\int_0^{\frac{1}{2}\pi} [\{K_0(\rho)\sec\theta\}^2 + \{K_1(\rho)\tan\theta\}^2]\,\mathrm{d}\theta, \tag{7.25}$$

where

$$\rho = (\rho_a + \tfrac{1}{2}\lambda)\sec\theta.$$

The resulting values agree with those obtained from equation (7.24).

(i) *Discussion*

The single, circular, flexible disc is the only stable configuration for a coalescence of numerous small, equal objects with negligible resistance to in-surface shearing forces. The forces of mutual attraction between two such discs would generally alter the disc shapes, and the process of coalescence would extend from an initial contact to the single, stable, circular shape as discussed in § 7 *c*. The analysis of § 7 *d* is, however, valid for discs with negligible out-of-surface rigidity provided there is adequate in-surface rigidity.

8. Experimental results

The experiments described below were conducted on distilled water, approximately 35 mm deep, in a shallow tank (500 mm by 500 mm by 75 mm). The surface of the water was maintained in a standard, clean state and the temperature was maintained at 26 °C so that $S = 0.07185\ \mathrm{N\,m^{-1}}$ and hence $L = 2.706$ mm. (It may be helpful to note that $1\ \mathrm{N\,m^{-1}} \approx 1.019\ \mathrm{g(wt)\,cm^{-1}}$.)

(*a*) *Force of mutual attraction*

This experiment was to measure the force of mutual attraction between two touching brass strips of thickness 0.46 mm, width 10 mm and length 125 mm. Each strip weighed 4.65 g which ensured that the water surface coincided with the upper edges of the strips, and hence the density of the strips, namely 8.09, had an effective value of 7.09, as per § 2 *e*. Ideally, one would like to determine the force required to separate the strips by applying equal and opposite, uniformly distributed, horizontal, 'opening' forces along the line of contact but this is not feasible without seriously affecting the phenomenon under investigation. Fortunately, the strips are sufficiently rigid for the total force of mutual attraction $\mathcal{H}$, say, to be determined with some accuracy from its equilibrium with equal and opposite 'opening' forces $\mathcal{H}^*$, say, applied at a single point along the line of contact. If this force system is applied at the mid-point of the line of contact, then $\mathcal{H} = \mathcal{H}^*$, but the experiments are much easier to conduct if this force system is applied at one end; then, by taking moments about the other end, $\mathcal{H} = 2\mathcal{H}^*$.

The experimental apparatus consisted of two parts: a torsion balance modified to measure the force applied to one strip, and a 'force applicator' that applied a force to the other strip. The torsion balance had a full-scale deflexion of 500 mg(wt) marked off in units of 1 mg. A coupling device converted horizontal forces experienced in the plane of the water surface into vertical forces on the torsion balance. The 'force applicator' consisted of a fine elastic thread lightly stretched between the outer ends of two watch springs, one in the water, the other above the surface. Each strip had a V-shaped notch, 1 mm wide by 1 mm deep, cut into one of its long edges, 1 mm from one end. These notches were arranged to be in register, so that they formed a hole which closed around the 'force applicator' and the tip of the torsion balance coupling device. The combined elasticity of the elastic thread and the watch springs made it possible to apply small force increments smoothly. A small force, insufficient to cause separation, was then applied to the strips by using the torsion balance in a conceptually reversed manner. Normally, the balance is subjected to an unknown force (e.g. a weight to be measured) whose value is then read off the dial by zeroing the balance with the dial arm. In this application a 'prescribed' force reading is set up on the dial with the dial arm and the balance zeroed by subjecting it to a force of that value. This is done by increasing the pull exerted by the 'force applicator'. A succession of increasing forces was now applied to the strips by the two-fold process of setting the torsion balance to a selected dial reading and then adjusting the balance indicator to zero by moving the 'force applicator'. By gradually increasing the force selected, a value was reached at which the strips finally separated before the balance was zeroed. The force selected was thus higher than that actually required whereas the previously chosen force was too low. A series of repeat experiments was then conducted with increasingly smaller increments towards the end. The horizontal force required to separate the two strips, namely $\mathcal{H}^* = 391$ mg, was thus determined by an iterative process. The corresponding value of $\mathcal{H}$ is accordingly 782 mg, but to compare this value with the theoretical prediction it is preferable to make allowance for the fact that the experimental strips are of finite length so that there is some support provided by surface tension at the ends. This can be done in an approximate way as follows. We denote by ϕ_{e} a surface slope normal to the ends of a strip and we assume that the deflexion ζ_{e}, say, along this normal decays as $\mathrm{e}^{-\nu}$ from its initial value ζ_{e}^*, say. It follows from arguments identical to that in § 2 *d* and elsewhere that

$$\phi_{\mathrm{e}} = \zeta_{\mathrm{e}}^*. \tag{8.1}$$

Now ζ_{e}^* varies linearly across the width from ζ_1^* to ζ_2^* and hence ϕ_{e} varies linearly from $\phi_{\mathrm{e},1}$ to $\phi_{\mathrm{e},2}$ where

$$\phi_{\mathrm{e},1} = \zeta_1^* \quad \text{and} \quad \phi_{\mathrm{e},2} = \zeta_2^*. \tag{8.2}$$

If we introduce l_s, the length of the strip, the equation of vertical equilibrium can now be expressed as

$$l_s S(\phi_1 + \phi_2) + 2bS(\phi_{e,1} + \phi_{e,2}) + \gamma g b l_s (z_1^* + z_2^*) = l_s W, \tag{8.3}$$

which, after substitution of equation (8.2), can be expressed non-dimensionally in the form

$$\phi_1 + \phi_2 + \beta(1+\delta)(\zeta_1^* + \zeta_2^*) = W/S, \tag{8.4}$$

where $\delta = 2L/l_s$. Likewise, the equation of moment equilibrium is given by

$$\beta(\phi_1 - \phi_2) + \{1 + \tfrac{1}{3}\beta^2(1+\delta)\}(\zeta_1^* - \zeta_2^*) = 0. \tag{8.5}$$

For the case of two touching strips

$$\phi_1 = 0 \quad \text{and} \quad \phi_2 = \zeta_2^*, \tag{8.6}$$

and equations (2.18), (8.4) and (8.5) yield

$$\mathcal{H}_{12}/l_s S = \tfrac{1}{2}{\zeta_1^*}^2, \tag{8.7}$$

where

$$\zeta_1^* = \frac{W}{S}\left(\frac{3 + 3\beta + \beta^2(1+\delta)}{3 + (6\beta + 4\beta^2)(1+\delta) + 2\beta^3(1+\delta)^2}\right). \tag{8.8}$$

The strips are such that $\beta = 1.848$ and $\delta = 0.0433$. The effective weight W per unit length of strip is $0.320\ \mathrm{N\,m^{-1}}$ and hence equations (8.7), (8.8) give

$$\begin{aligned}\mathcal{H}_{12} &= 7.21\ \mathrm{mN},\\ &= 737\ \mathrm{mg(wt)}.\end{aligned} \tag{8.9}$$

Agreement with the experimentally determined value, namely 782 mg, is slightly worse than this would indicate because equation (8.9) is based on linear theory, whereas from figure 6, $\mathcal{H}_{\text{exact}} \approx 1.15\mathcal{H}_{\text{linear}}$. The experimentally determined value of $\mathcal{H}$ thus underestimates the theoretical value by about 8%. This is not unexpected because, although the strips are effectively rigid in the plane normal to their thickness, they are susceptible to bending deformations out of this plane. Because the strips are tilted, the horizontally applied load system to which they are subjected — namely, the concentrated force $\mathcal{H}^*$ which is equilibrated by the uniformly distributed forces of mutual attraction — has a component normal to the surface of the strip. This component has an 'unzipping' action on the two strips that can only result in a lower measured value of $\mathcal{H}$.

(*b*) *Strips with localized repulsion*

For the experiments two metal strips, one of brass, the other of aluminium, were made from rolled sheet material. The aluminium strip was milled to the same thickness as that of the brass (0.45 mm) and then annealed to relieve the stress and reflatten it. The edges of both strips were milled to final size (280 mm long, 26.7 mm wide) in order to ensure right-angled corners. The brass strip (28.2 g) was 3.2 times heavier than the aluminium (8.8 g). The two strips were arranged side by side on a wire frame and gently lowered onto the water surface. The edges of the aluminium strip had been lightly rubbed against wax to ensure that they were hydrophobic and, as a result — and in contrast to the brass strip — the water surface left the aluminium strip along its lower edges. As soon as the strips were 'floating' freely,

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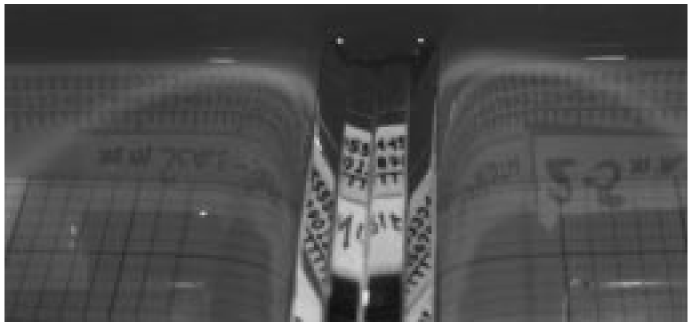
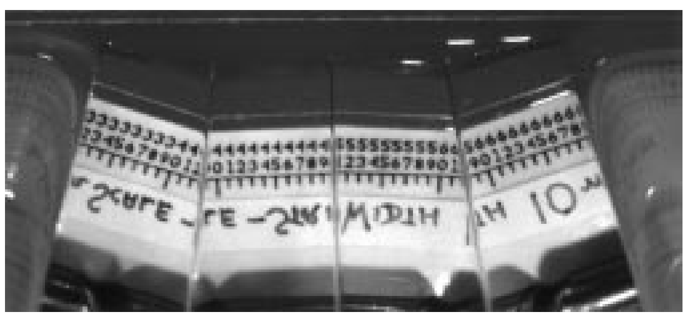


strip width 2.5 mm

strip width 6.5 mm



strip width 10 mm

Figure 14. Distinctive deflexion patterns in rafts with four strips whose widths correspond with those in figure 11.

they moved to an equilibrium position 0.90 mm apart, either separating or coming closer together depending on whether they were 'launched' touching one another or separated by a distance in excess of 0.90 mm. To compare this value with the prediction of linear theory we take account of the finite length of the strips by the adoption of equations (8.4), (8.5). An analysis similar to that of § 4 *c* then yields

$$H = \frac{U'V'\mathrm{e}^{-\lambda}(W_1U' - W_2P_1\mathrm{e}^{-\lambda})(W_2V' - W_1P_2\mathrm{e}^{-\lambda})}{2SQ(U'V' - P_1P_2\mathrm{e}^{-2\lambda})^2}, \tag{8.10}$$

where

$$\begin{aligned} U' &= \{1+\beta_2(1+\delta)\}\{3+3\beta_1+\beta_1^2(1+\delta)\}, \\ V' &= \{1+\beta_1(1+\delta)\}\{3+3\beta_2+\beta_2^2(1+\delta)\}, \\ P_1 &= \beta_1\{\beta_1^2(1+\delta)^2+3\delta\}, \\ P_2 &= \beta_2\{\beta_2^2(1+\delta)^2+3\delta\}, \\ Q &= \{1+\beta_1(1+\delta)\}\{1+\beta_2(1+\delta)\}. \end{aligned}$$

Corresponding to equation (4.24) we now have

$$[\lambda]_{\text{equilibrium}} = \ln\left(\frac{W_2 P_1}{W_1 U'}\right). \tag{8.11}$$

Both strips are such that $\beta = 4.933$ and $\delta = 0.0193$, while $W_{\text{aluminium}} = 0.3084\ \text{N m}^{-1}$ and $W_{brass} = 0.8662\ \text{N m}^{-1}$ with allowance for the water surface leaving at the upper edges. It follows that equation (8.11) predicts a gap of 0.846 mm.

(*c*) *Deflexion patterns in 4-strip rafts*

Three sets of four brass strips, all 125 mm long, were cut from 0.5 mm rolled sheet. The four strips in each set were of the same width, namely 2.5 mm, 6.5 mm or 10 mm, where 6.5 mm corresponds to $\beta = (3/2)^{1/2}$. The strips were cleaned and polished with Duraglit. The four strips of each set were then arranged side by side on a horizontal wire frame and gently lowered onto the surface of the water in the middle of the tank where they immediately assumed the distinctive deflexion patterns corresponding to those in figure 11. In order to make these patterns more apparent, each raft was moved under a low bridge on which was marked a millimetre scale with its numbers in reversed 'mirror writing' which are reflected across the succession of raft strips as shown in figure 14.

9. Conclusions

A theory is presented for determining the forces of mutual attraction — and, in certain cases, mutual repulsion — between objects supported primarily by surface tension. The key to this is the derivation of the equations of vertical equilibrium and moment equilibrium about orthogonal, horizontal axes for an arbitrarily shaped object at whose junction with the liquid and gas the free surface has arbitrary, but small, deflexions and slopes; see equations (3.19)–(3.21). When applied to all the objects and to any surrounding boundary, these equilibrium equations, which must also take into account the physical characteristics of the object, liquid and gas, provide the necessary boundary conditions for determining the surface deflexion of the liquid based on Laplace's theory. The horizontal forces required to maintain the static equilibrium of each object are now given by certain contour integrals involving the squares of the surface deflexion and slopes along the contact line between object, liquid and gas; see equations (3.22)–(3.24). In this connection we also present the theory for the transfer of static, non-uniform, horizontal forces through a liquid; see equations (3.28)–(3.30). This shows that the aforementioned contour integrals may be extended to include arbitrary regions of adjacent free surface of the liquid, thus opening up the possibility of a simpler evaluation. Thus, although the underlying theory is linear, it enables the prediction of the non-linearities associated with the

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forces of mutual attraction or repulsion. For example, the force $\mathcal{H}$ of mutual attraction between mirror-image objects is proportional to (object weight)^2. In general, however, the variation of $\mathcal{H}$ with object weights is more complex because it depends on the geometry and relative weight of each object. The variation of $\mathcal{H}$ with surface tension S also depends on these features, but an increase in S generally causes a decrease in $\mathcal{H}$; indeed, for objects where the proportion of support from hydrostatic pressure is negligible in comparison with that from surface tension, $\mathcal{H}$ varies inversely as S. For large values of the gap λ between two objects of finite size, where λ is in units of the capillary length L, the force $\mathcal{H}$ is proportional to the product of the objects' weights and decays asymptotically as $\lambda^{-1/2}\mathrm{e}^{-\lambda}$; for infinite strips $\mathcal{H}$ decays as $\mathrm{e}^{-\lambda}$. The theory can also predict the toppling instability of an object but it cannot predict the precise conditions at which an object sinks, because the surface slopes are then outside the range of linear theory; it can, however, predict a 'safe' weight of an object that is known to be somewhat less than the critical weight.

Certain one-dimensional cases of infinitely long, cylindrical strips supported on an infinite expanse of liquid have been considered because they admit of exact, large-deflexion solutions. These indicate that the deflexion of an object will be adequately given by linear theory even if the maximum slope of the adjacent liquid surface is as high as 45°, but for comparable accuracy in predicting the force of mutual attraction or repulsion, the maximum slope should not exceed about 25°. Other one-dimensional cases have been analysed by linear theory because they also demonstrate in a simple manner many of the phenomena associated with objects supported primarily by surface tension. Localized mutual repulsion can occur between two strips that are in close proximity if there are marked differences in their weights per unit area; its range is of the order of L and the equilibrium state is at the limit of this range, where $\mathcal{H}$ changes from repulsion to attraction. Mutual attraction or repulsion, which may be localized or extensive, may occur if a strip is near a boundary where the surface deflexion or slope is specified; indeed, for a range of specified slopes, there exists an unstable position of equilibrium such that the strip will coalesce with the boundary if displaced towards it, or move further away if displaced away from it.

Rafts of equal strips of shallow, rectangular section, held together by the effects of surface tension, exhibit characteristic deflexion patterns. If the width of each strip is $6^{1/2}L$, the horizontal forces that hold the raft together are provided solely by the edge strips that are tilted, while the central strips deflect an equal amount because they are supported solely by hydrostatic pressure; for narrower strips, successive deflexions of the junctions between strips increase monotonically towards the centre, as do the compressive forces across junctions; for wider strips, successive junction deflexions and compressive forces oscillate by decreasing amounts towards the centre.

We also determine the forces of mutual attraction or repulsion between two infinite strips with pie-crust edge undulations that cause a repetitive two-dimensional pattern of surface deflexion between the strips. Unless the undulations of the facing edges of the strips are identical and in mirror-image alignment, the strips will not coalesce and there is localized mutual repulsion. If the facing undulations are identical but not necessarily in alignment, shearing forces due to surface tension effects will ensure their alignment and so enable the strips to coalesce, at which point the force of mutual attraction can be markedly greater than for strips without edge undulations. The above results are unaffected by the presence or absence of undulations at the outer edges.

As for objects of finite size, the simplest to analyse are those with a flat lower surface whose boundary meets the free surface of the liquid. Within this category we determine the equilibrium state of an upright, circular cylinder with radially offset centre of gravity, and the toppling instability of an upright, homogeneous, circular cylinder. We also outline an inverse method of analysis for determining the force of mutual attraction between touching pairs of discs, and the method is demonstrated for discs of oval shape.

Finally we consider 'flexible discs' comprising coalescences of numerous objects whose individual linear dimensions are small in comparison with L. The forces of mutual attraction are determined within a circular, flexible disc, and between two dissimilar, circular, flexible discs.

Experimental confirmation of some of these theoretical results is presented, including the force of mutual attraction, an example of mutual repulsion and the distinctive deflexion patterns exhibited by rafts of strips.

We hope the theory may find application in branches of the physical and biological sciences.

When this paper was started, H.R.S. was at the Department of Materials Science and Engineering, University of Surrey.

Appendix A. Generalization of principle of Archimedes

If we apply Gauss's theorem to the deflexion function $\zeta(\xi,\eta)$ for the case of a single object supported on an infinite expanse of liquid, we have

$$-\oint \frac{\partial \zeta}{\partial \nu}\,\mathrm{d}\sigma = \iint \nabla^2 \zeta \,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{A 1}$$

where the area integration extends over the free surface of the liquid. It follows from equations (2.15), (2.19) that the equation of vertical equilibrium can also be expressed in the form

$$\iint \zeta \,\mathrm{d}\xi\,\mathrm{d}\eta + \mathcal{V} = \mathcal{W}. \tag{A 2}$$

Thus Archimedes' principle maintains its validity when surface tension effects are present, because the sum of the terms on the left of equation (A 2) is the total displaced liquid. Note that a more direct way of deriving equation (A 2) is to define the 'object' as the actual object plus the vanishingly thin, free surface layer to which purely horizontal forces are applied at infinity. The aforementioned sum of terms is then identified as the total hydrostatic upthrust on the 'object'. Furthermore, and in contrast to equation (3.19), it is clear that equation (A 2) maintains its validity in the exact, nonlinear regime.

Appendix B. Exact analysis for dissimilar strips

In what follows we adopt the notation of § 4 c and figure 8. The surface deflexion beyond the edges a and d are such that the arguments leading up to equation (2.21) remain valid, as does the second of equation (2.23):

$$\cos\phi_i = 1 - \tfrac{1}{2}{\zeta_i^*}^2, \qquad i = a, d. \tag{B 1}$$

Now, from equation (2.13), the vanishing of H for each strip implies that equation (B 1) is also valid for $i = b, c$. Indeed, this result stems from the vanishing of $\mathcal{N}_\xi$ in an exact (one-dimensional) formulation of equation (3.28), namely

$$\mathcal{N}_\xi = \tfrac{1}{2}\zeta^2 + \cos\{\arctan(\partial\zeta/\partial\xi)\} - 1. \tag{B 2}$$

The vanishing of $\mathcal{N}_\xi$ means that equation (B 2) is identical to equation (2.21) apart from a change of sign in the radical because ϕ_b is negative. This may be integrated to give,

$$\xi = (4-\zeta^2)^{1/2} - \ln\left(\frac{2+(4-\zeta^2)^{1/2}}{\zeta}\right) + \text{const.} \tag{B 3}$$

The equilibrium gap is now given in terms of the edge deflexions ζ_b^* and ζ_c^*, i.e.

$$[\lambda]_{\text{equilibrium}} = \xi(\zeta_c^*) - \xi(\zeta_b^*), \tag{B 4}$$

where ζ_b^*, ζ_c^* are determined from the equilibrium equations (2.11) and (2.14) for each strip, i.e.

$$\left.\begin{aligned} \sin\phi_a + \sin\phi_b + \beta_1(\zeta_a^* + \zeta_b^*)\cos\alpha_1 &= W_1/S, \\ \sin\alpha_1(\cos\phi_a + \cos\phi_b) + \cos\alpha_1(\sin\phi_a - \sin\phi_b) + \tfrac{1}{3}\beta_1(\zeta_a^* - \zeta_b^*) &= 0, \end{aligned}\right\} \tag{B 5}$$

and

$$\left.\begin{aligned} \sin\phi_c + \sin\phi_d + \beta_2(\zeta_c^* + \zeta_d^*)\cos\alpha_2 &= W_2/S, \\ \sin\alpha_2(\cos\phi_c + \cos\phi_d) + \cos\alpha_2(\sin\phi_c - \sin\phi_d) + \tfrac{1}{3}\beta_2(\zeta_c^* - \zeta_d^*) &= 0. \end{aligned}\right\} \tag{B 6}$$

The terms involving $\cos\phi_i$ are given by equation (B 1), while those involving $\sin\phi_i$ are given by equation (2.23) with due allowance for the reversals in the direction of ν, and with the assumption that strip_2 is heavier than strip_1, as in § 4 *c*. Thus

$$\left.\begin{aligned} \sin\phi_i &= \zeta_i^*(1 - \tfrac{1}{4}\zeta_i^{*^2})^{1/2}, \qquad i = a, c, d, \\ \sin\phi_b &= -\zeta_b^*(1 - \tfrac{1}{4}\zeta_b^{*^2})^{1/2}, \end{aligned}\right\} \tag{B 7}$$

while from equation (2.12) the angles of tilt of the strips are given by

$$\sin\alpha_1 = \left(\frac{\zeta_a^* - \zeta_b^*}{2\beta_1}\right), \qquad \sin\alpha_2 = \left(\frac{\zeta_c^* - \zeta_d^*}{2\beta_2}\right). \tag{B 8}$$

The deflexions ζ_a^*, ζ_b^* are determined by equation (B 5) while equation (B 6) shows that the strip deflects according to equation (2.24) with $\zeta_c^* = \zeta_d^*$.

Appendix C. Numerical analysis for touching oval discs

Following the analysis of § 6 *c*, attention is focused on the right hand disc, see figures 12, 13, where ξ is measured from its centroid. Further, with simplification in mind, and because we are dealing with a shallow disc, we denote the now equal values of $\xi_{\text{cg},\mathcal{W}}$ and $\xi'_{\text{cg},\mathcal{W}}$ by ξ_{cg}.

(i) *Determination of* μ

As a preliminary to the satisfaction of equation (3.21) with ξ_{cg} zero, we need to determine the position of the centre of gravity of the disc relative to the origin O_1. Let $\bar{\xi}, \bar{\eta}$ be cartesian coordinates whose origin is at O_1, then if the centre of gravity is at $\bar{\xi}_{\text{cg}}$, we have

$$\xi = \bar{\xi} - \bar{\xi}_{\text{cg}}, \tag{C 1}$$

so that from equation (6.2),

$$\bar{\xi}_{\mathrm{cg}} = \frac{1}{\mathcal{A}} \int \bar{\xi} \, \mathrm{d}\mathcal{A}. \tag{C 2}$$

We now drop the suffix 1 and denote the polar coordinates of the right-hand disc by ρ_n^*, θ_n where n goes from zero to N and $\theta_n = n\pi/N$. In terms of the numerical values of ρ_n^*, noting that $\rho_0^* = \rho_N^*$, we can therefore write

$$\mathcal{A} \approx \frac{\pi}{N} \sum_{n=0}^{N-1} {\rho_n^*}^2. \tag{C 3}$$

Likewise we have

$$\int \bar{\xi} \, \mathrm{d}\mathcal{A} \approx \frac{2\pi}{N} \sum_{n=1}^{N-1} {\rho_n^*}^2 \cos\theta_n \sin\theta_n \left\{ \rho_n^* \sin\theta_n - \frac{N}{2\pi}(\rho_{n+1}^* - \rho_{n-1}^*) \cos\theta_n \right\}. \tag{C 4}$$

Equations (C 3) and (C 4) determine $\bar{\xi}_{\mathrm{cg}}$ from (C 2) and hence the ξ, η axes are defined and we note that

$$\zeta_0 = \tfrac{1}{2}\{\zeta_C + \zeta_D + \bar{\xi}_{\mathrm{cg}}(\zeta_D - \zeta_C)/\chi\}. \tag{C 5}$$

Referring now to the various terms in equation (6.7), we have

$$\mathcal{I}_\eta = \mathcal{I}_{\bar{\eta}} - (\bar{\xi}_{\mathrm{cg}})^2 \mathcal{A}, \tag{C 6}$$

where $\mathcal{I}_{\bar{\eta}}$ is the second moment of area about the $\bar{\eta}$ axis, whence

$$\mathcal{I}_{\bar{\eta}} \approx \frac{2\pi}{N} \sum_{n=1}^{N-1} {\rho_n^*}^3 \cos^2\theta_n \sin\theta_n \left\{ \rho_n^* \sin\theta_n - \frac{N}{2\pi}(\rho_{n+1}^* - \rho_{n-1}^*) \cos\theta_n \right\}. \tag{C 7}$$

Also, when successive values of ϕ are determined we can write

$$\oint \xi^* \phi \, \mathrm{d}\sigma \approx \frac{\pi}{N} \left[\phi_0 \chi(\chi - \bar{\xi}_{\mathrm{cg}}) + 2 \sum_{n=1}^{N-1} \phi_n \rho^* \sec \Omega_n (\rho_n^* \cos\theta_n - \bar{\xi}_{\mathrm{cg}}) \right], \tag{C 8}$$

where Ω is the (clockwise) angle that an outward normal to the boundary makes with a radial vector from O_1, i.e.

$$\Omega = \arctan\left(\frac{1}{\rho^*} \frac{\partial \rho^*}{\partial \theta} \right), \text{ so that } \Omega_n \approx \arctan\left\{ \frac{N}{2\pi} \left(\frac{\rho_{n+1}^* - \rho_{n-1}^*}{\rho_n^*} \right) \right\}. \tag{C 9}$$

(ii) *Determination of* ϕ

The determination of ϕ is facilitated by the temporary re-introduction of suffices 1, 2 for the polar coordinates centred on O_1 and O_2. The radial and tangential components of the slopes at the disc boundary due to the deflexions ζ_1 and ζ_2 are then given by,

$$\left.\begin{aligned} Q_1,\ \text{say}, &= \left(\frac{\partial \zeta_1}{\partial \rho_1}\right)^* = -A\left[K_1(\rho_1^*) + \mu\left\{K_0(\rho_1^*) + \frac{1}{\rho_1^*}K_1(\rho_1^*)\right\}\cos\theta_1\right],\\ Q_2,\ \text{say}, &= \frac{1}{\rho_1^*}\left(\frac{\partial \zeta_1}{\partial \theta_1}\right)^* = -A\left[\frac{\mu}{\rho_1^*}K_1(\rho_1^*)\sin\theta_1\right],\\ Q_3,\ \text{say}, &= \left(\frac{\partial \zeta_2}{\partial \rho_2}\right)^* = -A\left[K_1(\rho_2^*) - \mu\left\{K_0(\rho_2^*) + \frac{1}{\rho_2^*}K_1(\rho_2^*)\right\}\cos\theta_2\right],\\ Q_4,\ \text{say}, &= \frac{1}{\rho_2^*}\left(\frac{\partial \zeta_2}{\partial \theta_2}\right)^* = A\left[\frac{\mu}{\rho_2^*}K_1(\rho_2^*)\sin\theta_2\right], \end{aligned}\right\} \tag{C 10}$$

where ρ_2^* and θ_2^* may be expressed in terms of ρ_1^*, θ_1^* via equation (6.13). Now, as noted prior to equation (C 9), Ω is the (clockwise) angle that an outward normal to the boundary makes with a radial vector from O_1. The corresponding value Ω', say, relative to the origin O_2 is given simply by

$$\Omega' = \Omega + \theta_2 - \theta_1. \tag{C 11}$$

It follows that

$$\left(\frac{\partial \zeta_1}{\partial \nu}\right)^* = Q_1\cos\Omega - Q_2\sin\Omega, \qquad \left(\frac{\partial \zeta_2}{\partial \nu}\right)^* = Q_3\cos\Omega' - Q_4\sin\Omega', \tag{C 12}$$

and hence

$$\phi = -\left(\frac{\partial \zeta_1}{\partial \nu}\right)^* - \left(\frac{\partial \zeta_2}{\partial \nu}\right)^*. \tag{C 13}$$

(iii) *Determination of μ and A*

We are now in a position to determine μ by trial-and-error for a given value of χ because the terms on the left-hand side of equation (3.21) are now known. Note that the value of μ is independent of the constant A which occurs there only as a common factor. The constant A in equation (6.12) is now given by equation (3.19) where we note that

$$\oint \phi\, \mathrm{d}\sigma \approx \frac{\pi}{N}\left[\phi_0\chi + 2\sum_{n=1}^{N-1}\phi_n\rho_n^*\sec\Omega_n\right]. \tag{C 14}$$

The values of μ and A given in table 2 were obtained by taking $N = 100$.

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Received 21 May 1996; accepted 6 August 1996